Problems in Linear Algebra
In preparing this book of problems the author attempted, firstly, to give a sufficient number of exercises for developing skills in the solution of typical problems (for example, the computing of determinants with numerical elements, the solution of systems of linear equations with numerical coefficients, and the like), secondly, to provide problems that will help to clarify basic concepts and their interrelations (for example, the connection between the properties of matrices and those of quadratic forms, on the one hand, and those of linear transformations, on the other), thirdly, to provide for a set of problems that might supplement the course of lectures and help to expand the mathematical horizon of the student (instances are the properties of the Pfaffian of the skew-symmetric determinant, the properties of associated matrices, and so on).

A number of problems involving the proof of theorems that can be found in textbooks are also given. These problems were included because the instructor often (for lack of time) gives part of the material as homework for the student on the basis of the textbook, and this can be done on the basis of the problem book where hints are given for working out proofs. The author feels this can help to develop habits of scientific investigation.

Compared with other problem books, this one has a few new basic features. They include problems dealing with polynomial matrices (Sec. 13), linear transformations of affine and metric spaces (Secs. 13 and 19), and a supplement devoted to groups, rings, and fields. The problems of the supplement deal with the most elementary portions of the theory. Still and all, I think it can be used in pre-seminar discussions in the first and second years of study.

The contents and the sequence of presentation of material in a lecture course depend largely on the lecturer. The author has tried to take into account this diversity of pro-
sentation, and the result has been a certain amount of duplication. For example, the same facts are given first in the section devoted to quadratic forms and then again in the chapter on linear transformations; some of the problems are stated so that they can be worked out in the case of a real Euclidean space and also in that of a complex unitary space. I believe that this makes for a certain flexibility of use of the problem book.

Some sections contain an introduction with a few definitions and a brief discussion of terminology and notation when the available textbooks do not exhibit complete unity in this respect. An exception is the introduction to Sec. 5, where basic methods presented for computing determinants of any order and examples of each method are given. This was done because such information is usually not given in standard textbooks, and students encounter considerable difficulties.

Starred numbers indicate problems that have been worked out or provided with hints. Solutions are given for a small number of problems. These are either problems involving a general method that is then applied to a series of other problems (for instance, problem 1151 which offers a method for computing the function of a matrix, problem 1529 which contains the construction of a basis in which the matrix of a linear transformation has Jordan form) or problems that are very difficult (say, problems 1433, 1614, 1617). As a rule, the hints contain only a suggestion of the idea or method of solution and leave to the student the actual solving. Only in the case of more difficult problems is a general plan of solution provided (see problems 546, 1492, 1632).

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Chapter I

Determinants

Sec. 1. Second- and Third-Order Determinants

Compute the following determinants:

1. $\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}$
2. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$
3. $\begin{vmatrix} 3 & 2 \\ 8 & 5 \end{vmatrix}$
4. $\begin{vmatrix} 6 & 9 \\ 8 & 12 \end{vmatrix}$
5. $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$
6. $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix}$
7. $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$
8. $\begin{vmatrix} a^2+ab+b^2 & a^2-ab+b^2 \\ a+b & a-b \end{vmatrix}$
9. $\begin{vmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{vmatrix}$
10. $\begin{vmatrix} \sin \alpha & \cos \alpha \\ \sin \beta & \cos \beta \end{vmatrix}$
11. $\begin{vmatrix} \cos \alpha & \sin \alpha \\ \sin \beta & \cos \beta \end{vmatrix}$
12. $\begin{vmatrix} \sin \alpha + \sin \beta & \cos \beta + \cos \alpha \\ \cos \beta - \cos \alpha & \sin \alpha - \sin \beta \end{vmatrix}$
13. $\begin{vmatrix} 2 \sin \varphi \cos \varphi & 2 \sin^2 \varphi - 1 \\ 2 \cos^2 \varphi - 1 & 2 \sin \varphi \cos \varphi \end{vmatrix}$
14. $\begin{vmatrix} 1-\tau^2 & 2t \\ \frac{1}{1+\tau^2} & \frac{1}{1+\tau^2} \end{vmatrix}$
15. $\begin{vmatrix} \frac{(1-\tau)^3}{1+\tau^2} & 2\tau \\ \frac{2\tau}{1+\tau^2} & -\frac{(1+\tau)^3}{1+\tau^2} \end{vmatrix}$
16. $\begin{vmatrix} \frac{1+\tau^2}{1-\tau^2} & \frac{2t}{1-\tau^3} \\ \frac{2t}{1-\tau^3} & \frac{1+\tau^2}{1-\tau^3} \end{vmatrix}$
17. $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}$

Compute the following determinants ($t = \sqrt{-1}$):

18. $\begin{vmatrix} a & c-di \\ c+di & b \end{vmatrix}$
19. $\begin{vmatrix} a+bi & b \\ c-di & b \end{vmatrix}$
20. \[ \begin{vmatrix} \cos \alpha + i \sin \alpha & 1 \\ 1 & \cos \alpha - i \sin \alpha \end{vmatrix} \]

21. \[ \begin{vmatrix} a + bi & c + di \\ -c + di & a - bi \end{vmatrix} \]

Use determinants to solve the following systems of equations:
22. \[ 2x + 5y = 1, \quad 3x + 7y = 2. \]
23. \[ 2x - 3y = 4, \quad 4x - 5y = 10. \]
24. \[ 5x - 7y = 1, \quad x - 2y = 0. \]
25. \[ 4x + 7y + 13 = 0, \quad 5x + 8y + 14 = 0. \]
26. \[ x \cos \alpha - y \sin \alpha = \cos \beta, \quad \sin \alpha + y \cos \alpha = \sin \beta. \]

Investigate to see whether the given system of equations is even determined (has a unique solution), indeterminate (has an infinity of solutions) or is inconsistent (no solution):
27. \[ x \tan \alpha + y = \sin (\alpha + \beta). \]
28. \[ x - y \tan \alpha = \cos (\alpha + \beta), \]

where \( \alpha \neq \frac{n \pi}{2} + kn \) (\( k \) an integer).

29. \[ 4x + 6y = 2, \quad (\text{Do Cramer's formulas yield a correct} \]
\[ 6x + 9y = 3. \quad \text{answer?}) \]
30. \[ 3x - 2y = 2, \quad (a - b) x = b - c. \]
31. \[ x \sin \alpha = 1 + \sin \alpha, \quad 6x - 4y = 3. \]
32. \[ x \sin \alpha = 1 + \cos \alpha. \]
33. \[ x \sin (\alpha + \beta) = \sin \alpha + \sin \beta. \]
34. \[ a^2 x = ab, \quad 34. \quad ax + by = ad, \]
\[ abx = b^2. \quad bx + cy = bd. \]
35. \[ ax + 4y = 2, \quad 36. \quad ax - 9y = 6, \]
\[ 9y + ay = 3. \quad 10x - by = 10. \]

37. \[ \text{Prove that for a determinant of the second order to be equal to zero it is necessary and sufficient that the rows are proportional. The same holds true for columns as well (if certain elements of the determinant are zero, the proportionality may be understood in the sense that the elements of one row are obtained from the corresponding elements of another row by multiplying by the same number, which may even be zero).} \]
61. \[
\begin{vmatrix}
\alpha^2 + 1 & \alpha \beta & \alpha \\
\alpha \beta & \beta^2 + 1 & \beta \\
\alpha & \beta & \gamma^2 + 1
\end{vmatrix}
\]

62. \[
\begin{vmatrix}
\cos \alpha & \sin \alpha \cos \beta & \sin \alpha \cos \beta \\
\sin \alpha \cos \beta & \cos \alpha \cos \beta & \cos \alpha \sin \beta \\
\cos \alpha \sin \beta & \sin \alpha \sin \beta & \cos \beta
\end{vmatrix}
\]

63. \[
\begin{vmatrix}
\sin \alpha & \cos \alpha & 1 \\
\sin \beta & \cos \beta & 1 \\
\sin \gamma & \cos \gamma & 1
\end{vmatrix}
\]

64. Determine under what condition the following equation holds true:
\[
\begin{vmatrix}
1 & \cos \alpha & \cos \beta \\
\cos \alpha & 1 & \cos \gamma \\
\cos \beta & \cos \gamma & 1
\end{vmatrix} = 0
\]

65. Show that the determinant
\[
\begin{vmatrix}
a^2 & b \sin \alpha & c \sin \alpha \\
b \sin \alpha & 1 & \cos \alpha \\
c \sin \alpha & \cos \alpha & 1
\end{vmatrix}
\]

and two other determinants obtained from this one by a circular permutation of the elements \(a, b, c\) and \(\alpha, \beta, \gamma\) are equal to zero if \(a, b, c\) are the lengths of the sides of a triangle and \(\alpha, \beta, \gamma\) are the angles opposite the sides \(a, b, c\), respectively.

Evaluate the following third-order determinants (\(i = \sqrt{-1}\)):

66. \[
\begin{vmatrix}
1 & 0 & 1 + i \\
0 & 1 & i \\
1 - i & -1 & 1
\end{vmatrix}
\]

67. \[
\begin{vmatrix}
x & a + bi & c + di \\
a - bi & y + e + fi \\
c - di & e - fi & z
\end{vmatrix}
\]

68. \[
\begin{vmatrix}
e & e^2 & 1 \\
e^2 & 1 & e \\
e & e^2 & 1
\end{vmatrix}, \text{ where } e = \frac{1}{2} + i \frac{\sqrt{3}}{2}
\]

69. \[
\begin{vmatrix}
i & e & 1 \\
i & e^2 & 1 \\
i & i e^2 & 1
\end{vmatrix}, \text{ where } e = \cos \frac{2}{3} \pi + i \sin \frac{2}{3} \pi.
\]

70. \[
\begin{vmatrix}
1 & 1 & 1 \\
e & e^2 & 1
\end{vmatrix}, \text{ where } e = \cos \frac{4}{3} \pi + i \sin \frac{4}{3} \pi.
\]

71. Prove that if all the elements of a third-order determinant are equal to \(\pm 1\), then the determinant itself will be an even number.

*72. Find the maximum value that can be assumed by a third-order determinant provided that all its elements are equal to \(\pm 1\).

*73. Find the largest value of a third-order determinant provided that the elements are equal to \(\pm 1\) or \(0\).

Use determinants to solve the following systems of equations:

74. \[
\begin{align*}
2x + 3y + 5z &= 10, \\
3x + 7y + 4z &= 3,
\end{align*}
\]

75. \[
\begin{align*}
5x - 6y + 4z &= 3, \\
x + 2y + 2z &= 3.
\end{align*}
\]

76. \[
\begin{align*}
4x - 3y + 2z + 4 &= 0, \\
6x - 2y + 3z + 1 &= 0,
\end{align*}
\]

77. \[
\begin{align*}
5x + 2y + 3z + 2 &= 0, \\
2x - 2y + 5z &= 0, \\
3x + 4y + 2z + 10 &= 0.
\end{align*}
\]

*78. \[
\begin{align*}
\frac{x}{a} - \frac{y}{b} + 2 &= 0, \\
-\frac{2y}{b} + \frac{3z}{c} - 4 &= 0,
\end{align*}
\]

79. \[
\begin{align*}
2ax - 3by + cz &= 0, \\
3ax - 6by + 5cz &= 2abc, \\
5ax - 4by + 2cz &= 3abc,
\end{align*}
\]

where \(abc \neq 0\).

*80. \[
\begin{align*}
4bcx + acy - 2abz &= 0, \\
5bcx + 3acy - 4abz + abc &= 0, \\
3bcx + 2acy - abz - 4abc &= 0 \quad (abc \neq 0).
\end{align*}
\]

*81. Solve the following system of equations:
\[
\begin{align*}
x + y + z &= a, \\
x + ey + e^2z &= b, \quad (e \text{ is a value of } \frac{3}{\sqrt{4}} \text{ different from } 1), \\
x + e^2y + ez &= c.
\end{align*}
\]
Investigate each system of equations to determine whether it is even determined, indeterminate, or inconsistent:

82. \(2x - 3y + z = 2\), \(3x - 5y + 6z = 5\).

83. \(4x + y + 2z = 1\), \(x + 3y + 5z = -1\), \(3x + 6y + 9z = 2\).

84. \(5x - 6y - z = 4\), \(2x - y + 3z = 4\), \(3x - 5y - 2z = 3\), \(2x - y + 3z = 5\).

85. \(4x - 3y - 2z = 1\), \(5x - 4y = -2\).

86. \(2ax - 23y + 29z = 4\), \(2x + ay + 4z = 7\), \(5x + 2y + az = 5\), \(9x - 7y + 8az = 0\).

87. \(ax + 2y + z = 0\), \(ax + 2y = 2\), \(2y + 3z - 1 = 0\), \(5x + 2y = 1\), \(3x - 2y + 2z = 0\).

88. \(ax + 4y + 2z = 0\).

Prove, either by direct computation via the triangle rule or by the rule of Sarrus, the following properties of third-order determinants:

90. A determinant of the third order remains unchanged if the rows and columns are interchanged (that is to say, if the matrix of the determinant is transposed).

91. If all elements of some row (or column) are equal to zero, then the determinant itself is equal to zero.

92. If all elements of some row (or column) of a determinant are multiplied by one and the same number, the whole determinant is then multiplied by that number.

93. A determinant changes sign if two rows (or two columns) are interchanged.

94. If two rows (or two columns) of a determinant are the same, the determinant is zero.

95. If all elements of one row are proportional to the corresponding elements of another row, the determinant is equal to zero (the same holds true for the columns).

96. If each element of some row of a determinant is represented as the sum of two terms, then the determinant is equal to the sum of two determinants in which all rows, except the given row, remain the same, while the given row in the first determinant contains the first terms and the given row in the second determinant contains the second terms (the same holds true for the columns).

97. A determinant remains unchanged if elements of one row of the determinant are added to the corresponding elements of another row multiplied by the same number (the same applies to columns).

98. We say that one row of a determinant is a linear combination of the other rows if each element of the given row is equal to the sum of the products of the corresponding elements of the other rows into certain numbers that are constant for each row, that is to say, that are independent of the position number of the element in the row. A similar definition applies to a linear combination of columns. For example, the third row of the determinant

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

is a linear combination of the first two if there exist numbers \(c_1\) and \(c_2\) such that \(a_{3j} = c_1a_{1j} + c_2a_{2j} (j = 1, 2, 3)\).

Prove that if one row (or column) of a third-order determinant is a linear combination of the other rows (or columns), the determinant is equal to zero.

Hint. The converse is also true, but it follows from a further development of the theory of determinants.

99. Use the preceding problem to demonstrate with an illustration that, unlike second-order determinants (see problem 38), the proportionality of two rows (or columns) is no longer necessary for a third-order determinant to be equal to zero.

Using the properties of third-order determinants given in problems 91-98, compute the following determinants:

100. \[\begin{vmatrix}
sin^2 \alpha & 1 & \cos^2 \alpha \\
\sin^2 \beta & 1 & \cos^2 \beta \\
\sin^2 \gamma & 1 & \cos^2 \gamma
\end{vmatrix}\]

101. \[\begin{vmatrix}
sin^2 \alpha & \cos 2\alpha & \cos^2 \alpha \\
\sin^2 \beta & \cos 2\beta & \cos^2 \beta \\
\sin^2 \gamma & \cos 2\gamma & \cos^2 \gamma
\end{vmatrix}\]

102. \[\begin{vmatrix}
x & x' & ax + bx' \\
y & y' & ay + by' \\
z & z' & az + bz'
\end{vmatrix}\]

103. \[\begin{vmatrix}
a_1 + b_1 & a_1 + b_1 & a_1b_1 \\
(a_1 + b_1) & a_1^2 + b_1^2 & a_1b_1 \\
(a_1 + b_1) & a_1^2 + b_1^2 & a_1b_1 \\
(a_1 + b_1) & a_1^2 + b_1^2 & a_1b_1
\end{vmatrix}\]

104. \[\begin{vmatrix}
a + b & a + c & a + b + c \\
a + b & a + c & a + b + c \\
a + b & a + c & a + b + c
\end{vmatrix}\]

105. \[\begin{vmatrix}
(a^2 + a - c)^2 & (a^2 - a + c)^2 \\
(b^2 + b - c)^2 & (a^2 - a + c)^2 \\
(c^2 + c - a)^2 & (a^2 - a + c)^2
\end{vmatrix}\]
106. \[ \begin{vmatrix} 1 & x & z^2 \\ x & y & z \end{vmatrix}, \text{ where } z \text{ is a value of } \sqrt{3} \text{ different from } 1. \]

107. \[ \begin{vmatrix} \sin \alpha & \cos \alpha & \sin (\alpha + \beta) \\ \sin \beta & \cos \beta & \sin (\beta + \gamma) \\ \sin \gamma & \cos \gamma & \sin (\gamma + \delta) \end{vmatrix} \]

108. \[ \begin{vmatrix} a_1 + b_1 i & a_1 i - b_1 c_1 \\ a_2 + b_2 i & a_2 i - b_2 c_2 \\ a_3 + b_3 i & a_3 i - b_3 c_3 \end{vmatrix}, \text{ where } i = \sqrt{-1}. \]

109. \[ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 + \lambda x_2 & y_1 + \lambda y_2 & 1+\lambda \end{vmatrix} \]

*110. \[ \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}, \text{ where } \alpha, \beta, \gamma \text{ are roots of the equation } x^3 + px + q = 0. \]

Prove the following identities without expanding the determinants:

111. \[ \begin{vmatrix} a_1 & b_1 & a_1 x + b_1 y + c_1 \\ a_2 & b_2 & a_2 x + b_2 y + c_2 \\ a_3 & b_3 & a_3 x + b_3 y + c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \]

112. \[ \begin{vmatrix} a_1 + b_1 x & a_1 - b_1 x & c_1 \\ a_2 + b_2 x & a_2 - b_2 x & c_2 \\ a_3 + b_3 x & a_3 - b_3 x & c_3 \end{vmatrix} = -2x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \]

113. \[ \begin{vmatrix} a_1 + b_1 i & a_1 i + b_1 c_1 \\ a_2 + b_2 i & a_2 i + b_2 c_2 \\ a_3 + b_3 i & a_3 i + b_3 c_3 \end{vmatrix} = 2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \]

114. \[ \begin{vmatrix} a_1 + b_1 x & a_1 x + b_1 c_1 \\ a_2 + b_2 x & a_2 x + b_2 c_2 \\ a_3 + b_3 x & a_3 x + b_3 c_3 \end{vmatrix} = \begin{vmatrix} 1 & a & b \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \]

115. \[ \begin{vmatrix} 1 & a & b \\ 1 & b & c \\ 1 & c & a \end{vmatrix} = (b - a)(c - a)(c - b). \]

Sec. 2. Permutations and Substitutions

Determine the number of inversions in the following permutations (unless otherwise stated, the initial arrangement is taken to be 1, 2, 3, ..., in increasing order):

123. 2, 3, 5, 4, 1. 124. 6, 3, 1, 2, 5, 4.
125. 1, 9, 6, 3, 2, 5, 4, 7, 8. 126. 7, 5, 6, 4, 1, 3, 2.
127. 1, 3, 5, 7, ..., 2n - 1, 2, 4, 6, 8, ..., 2n.
128. 2, 4, 6, ..., 2n, 1, 3, 5, ..., 2n - 1.

In the following permutations, determine the number of inversions and indicate the general criterion of those numbers \( n \) for which the permutation is even and those for which...
it is odd:

129. 4, 4, 7, ..., 3n - 2, 2, 5, 8, ..., 3n - 1, 3, 6, 9, ..., 3n.

130. 3, 6, 9, ..., 3n, 2, 5, 8, ..., 3n - 1, 1, 4, 7, ...

3n - 2.

131. 2, 5, 8, ..., 3n - 1, 3, 6, 9, ..., 3n, 1, 4, 7, ...

3n - 2.

132. 2, 5, 8, ..., 3n - 1, 1, 4, 7, ..., 3n - 2, 3, 6, 9, ..., 3n.

133. 1, 5, ..., 4n - 3, 2, 6, ..., 4n - 2, 3, 7, ...

4n - 1, 4, 8, ..., 4n.

134. 1, 5, ..., 4n - 3, 3, 7, ..., 4n - 1, 2, 6, ...

4n - 2, 4, 8, ..., 4n.

135. 4n, 4n - 4, ..., 8, 4, 4n - 1, 4n - 5, ..., 7, 3, 4n - 2, 4n - 6, ..., 6, 2, 4n - 3, 4n - 7, ..., 5, 1.

136. In what permutation of the numbers 1, 2, ..., n is the number of inversions greatest and what is it?

137. How many inversions does the number 1 in the kth position of a permutation generate?

138. How many inversions are generated by the number n in the kth position in a permutation of the numbers 1, 2, 3, ..., n?

139. What is the sum of the number of inversions and the number of order arrangements in any permutation of the numbers 1, 2, ..., n?

140. For what numbers n is the parity of the number of inversions and the number of arrangements in all permutations of the numbers 1, 2, ..., n the same and for what numbers is it different?

141. Prove that the number of inversions in the permutation $a_1, a_2, ..., a_n$ is equal to the number of inversions in the permutation of the indices 1, 2, ..., n obtained if the given permutation is replaced by the original arrangement.

142. Show that it is possible to pass from one permutation $a_1, a_2, ..., a_n$ to another permutation $b_1, b_2, ..., b_n$ of the same elements via at most $n - 1$ transpositions.

143. Give an illustration of a permutation of the numbers 1, 2, 3, ..., n which cannot be brought to a normal arrangement in less than $n - 1$ transpositions, and prove this.

*144. Prove that it is possible to pass from one permutation $a_1, a_2, ..., a_n$ to any other permutation $b_1, b_2, ..., b_n$ of the same elements via at most $\frac{n(n-1)}{2}$ adjacent transpositions (transpositions of adjacent elements, that is).

*145. Given: the number of inversions in the permutation $a_1, a_2, ..., a_{n-1}, a_n$ is equal to $k$. How many inversions are there in the permutation $a_1, a_{n-1}, ..., a_2, a_n$?

*146. How many inversions are there in all permutations of $n$ elements taken together?

*147. Prove that it is possible to pass from any permutation of the numbers 1, 2, ..., n, containing $k$ inversions, to the original arrangement by means of $k$ adjacent transpositions, but it is impossible to do so via a smaller number of such transpositions.

*148. Prove that for any integer $k$ ($0 \leq k \leq C_n^a$) there exists a permutation of the numbers 1, 2, ..., n, the number of inversions of which is equal to $k$.

*149. Denote by $(n, k)$ the number of permutations of the numbers 1, 2, ..., n, each of which contains exactly $k$ inversions. For the number $(n, k)$ derive the recurrence relation

$$(n + 1, k) = (n, k) + (n, k - 1) + (n, k - 2) + \cdots + (n, k - n),$$

where $(n, j)$ must be 0 for $j > C_n^a$ and for $j < 0$. Use this relation to set up an array of the numbers $(n, k)$ for $n = 1, 2, 3, 4, 5, 6$ and $k = 0, 1, 2, \ldots, 15$.

*150. Prove that the number of permutations of the numbers 1, 2, ..., n containing $k$ inversions is equal to the number of permutations of the same numbers containing $C_n^a - k$ inversions.

Expand the following substitutions into products of independent cycles and determine their parity by the decrement (that is, the difference between the number of actually permuted elements and the number of cycles). To simplify counting the decrement, one-term cycles can be introduced into the expansion for numbers that remain in place.

151. (1 2 3 4 5) 152. (1 2 3 4 5 6)

153. (4 1 5 2 3) 154. (5 4 1 4 2 3)

155. (1 2 3 4 5 6 7 8)
154. \((1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)\) \((5\ 8\ 9\ 2\ 1\ 4\ 3\ 6\ 7)\)  
155. \((1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)\) \((2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 1)\)  
156. \((1\ 2\ 3\ 4\ 5\ 6)\) \((4\ 5\ 6\ 1\ 2\ 3)\)  
157. \((1, 2, 3, 4, \ldots, 2n-1, 2n)\) \((2, 1, 4, 3, \ldots, 2n, 2n-1)\)  
158. \((1, 2, 3, 4, 5, 6, \ldots, 3n-2, 3n-1, 3n)\) \((3, 2, 1, 6, 5, 4, \ldots, 3n, 3n-1, 3n-2)\)  
159. \((1, 2, 3, 4, \ldots, 2n-3, 2n-2, 2n-1, 2n)\) \((3, 4, 5, 6, \ldots, 2n-1, 2n, 1, 2)\)  
160. \((1, 2, 3, 4, 5, 6, \ldots, 3n-2, 3n-1, 3n)\) \((2, 3, 1, 5, 6, 4, \ldots, 3n-1, 3n, 3n-2)\)  
161. \((1, 2, 3, 4, 5, 6, \ldots, 3n-2, 3n-1, 3n)\) \((4, 5, 6, 7, 8, 9, \ldots, 1, 2, 3)\)  
162. \((1, 2, \ldots, k, \ldots, nk-k+1, nk-k+2, \ldots, nk)\) \((k+1, k+2, \ldots, 2k, \ldots, 1, 2, \ldots, k)\)  

In the following permutations, change from cyclic notation to two-row notation:

163. \((1\ 5)\ (2\ 3\ 4)\) 164. \((1\ 3)\ (2\ 5)\ (4)\)  
165. \((7\ 5\ 3\ 1)\ (2\ 4\ 6)\ (8)\ (9)\)  
166. \((1\ 2)\ (3\ 4)\ \ldots\ (2n-1, 2n)\)  
167. \((1\ 2\ 3\ 4\ \ldots\ 2n-1, 2n)\)  
168. \((3\ 2\ 1)\ (6\ 5\ 4)\ \ldots\ (3n, 3n-1, 3n-2)\)  

Multiply the following substitutions:

169. \[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 1 & 3 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
3 & 2 & 4 & 1
\end{pmatrix}
\]
170. \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 4 & 5 & 1 & 3
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
5 & 3 & 4 & 1 & 2
\end{pmatrix}
\]
171. \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 5 & 1 & 2 & 4
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 4 & 1 & 5 & 2
\end{pmatrix}
\]
172. \[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1
\end{pmatrix}^2
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 4 & 5 & 1 & 2
\end{pmatrix}^3
\]
173. \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 4 & 5 & 1 & 2
\end{pmatrix}^3
\]

174. Prove that if the power of a cycle is unity, then the exponent of the power is divisible by the length of the cycle. (The length of a cycle is the number of its elements.)

175. Prove that from among all powers of a substitution equal to unity the smallest exponent is equal to the smallest common multiple of the lengths of cycles that enter into the decomposition of the substitution.

*176. Find \(A^{100}\), where \(A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}\)

177. Find \(A^{100}\), where \(A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}\)

178. Find the substitution \(X\) from the equation \(AXB = C\), where

\[
A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 2 & 4 & 1 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 7 & 4 & 5 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 6 & 4 & 7 & 2 \end{pmatrix}
\]

179. Prove that premultiplication of a substitution by a transposition (that is, a two-member cycle) \((\alpha, \beta)\) is equivalent to a transposition (an interchange) of the number \(\alpha\) and \(\beta\) in the upper row of the substitution, and postmultiplication by the same transposition is equivalent to a transposition of \(\alpha\) and \(\beta\) in the lower row of the substitution.

180. Prove that if the numbers \(\alpha\) and \(\beta\) enter into one cycle of a substitution, then in the case of pre- or postmultiplication of the substitution by the transposition \((\alpha, \beta)\), the given cycle decomposes into two cycles, but if the numbers \(\alpha\) and \(\beta\) appear in different cycles, then these cycles merge into one under the given multiplication.

181. Use the two preceding problems to prove that the number of inversions and the decrement of any substitution have the same parity.

*182. Prove that the smallest number of transpositions into the product of which the given substitution is decomposed is equal to its decrement.

*183. Prove that the smallest number of transpositions that carry the permutation \(a_1, a_2, \ldots, a_n\) into the permutation \(b_1, b_2, \ldots, b_n\) equals the number of inversions in the permutation.
tion $b_1, b_2, \ldots, b_n$ of the same elements is equal to the decrement of the substitution

$$p = \begin{pmatrix} a_1, a_2, \ldots, a_n \\ b_1, b_2, \ldots, b_n \end{pmatrix}.$$  

184. Find all substitutions of the numbers 1, 2, 3, 4 that are permutable with the substitution

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 \end{pmatrix}.$$  

185. Find all substitutions of the numbers 1, 2, 3, 4, 5 that are permutable with the substitution

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}.$$  

186. For any two integers $x$ and $m$, where $m \neq 0$, denote by $r(x, m)$ the remainder (assumed to be nonnegative) obtained from the division of $x$ by $m$. Prove that if $m \geq 2$ and $a$ is an integer prime to $m$, then the correspondence $x \rightarrow (a, m, x = 1, 2, \ldots, m - 1)$ is a substitution of the numbers 1, 2, ..., $m - 1$.

187. Write the substitution of the numbers 1, 2, 3, 4, 5, 6, 7, 8 under which the number $x$ passes into the remainder obtained from the division of $5x$ by 9.

Sec. 3. Definition and Elementary Properties of Determinants of any Order

The problems of this section are aimed at elucidating the concept of a determinant of any order and its most elementary properties, including that of a determinant being equal to zero when the rows are linearly dependent, and the expansion of a determinant in terms of a row.

Problems intended to develop skill in computing determinants with numerical elements, problems involving methods of computing determinants of a special type or dealing with the Laplace theorem, and problems dealing with the multiplication of determinants are all dealt with in the section that follows Sec. 3.

Determine which of the products given below enter into determinants of appropriate orders and with what signs:

188. $a_{43}a_{31}a_{35}a_{12}a_{54}$. 189. $a_{31}a_{33}a_{35}a_{33}a_{23}a_{54}$.
190. $a_{47}a_{51}a_{45}a_{25}a_{43}a_{42}$. 191. $a_{33}a_{10}a_{25}a_{45}a_{52}a_{51}a_{44}$.
192. $a_{12}a_{23}a_{54} \ldots a_{n-1}, n \times n$. 193. $a_{13}a_{23} \ldots a_{n-1}, n \times n$.
194. $a_{42}a_{31}a_{34}a_{14} \ldots a_{2n-1}, 2n \times 2n$. 195. $a_{41}a_{22}, n \times n-1 \ldots a_{n}, 2$.
196. $a_{13}a_{33}a_{45}a_{52}a_{44} \ldots a_{3n-2}, 3n \times 3n-1, 3n \times 3n-2$.
197. Choose the values of $i$ and $k$ so that the product

$$a_{22}a_{33}a_{44}a_{52}a_{61}$$

appears in a sixth-order determinant with the minus sign.

198. Choose the values of $i$ and $k$ so that the product

$$a_{47}a_{51}a_{45}a_{25}a_{43}a_{42}$$

enters into a seventh-order determinant with the plus sign.

199. Find terms of a fourth-order determinant containing the element $a_{32}$ and appearing in the determinant with the plus sign.

200. Find terms of the determinant

$$\begin{vmatrix} 5x & 1 & 2 & 3 \\ x & x & 1 & 2 \\ 3 & x & 1 & 2 \end{vmatrix}$$

that contain $x^4$ and $x^5$.

201. What is the sign of a determinant of order $n$ in a product of elements of the principal diagonal?

202. What sign does a determinant of order $n$ have in a product of elements of the secondary diagonal?

203. Using only the definition of a determinant, compute

$$\begin{vmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 \\ a_{31} & a_{32} & a_{33} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix}$$

where all elements above or below the principal diagonal are zero.
204. Using only the definition of a determinant, compute
\[
\begin{vmatrix}
0 & 0 & 0 & a_{1n} \\
0 & a_{2,n-1} & a_{2n} \\
& & \ddots & \ddots \\
& & & a_{n-2,n-2} & a_{n-2,n} \\
a_{n1} & \cdots & a_{n,n-2} & a_{n,n-1} & a_{nn}
\end{vmatrix}
\]
where all elements above or below the secondary diagonal are zero.

205. Using only the definition of a determinant, compute
\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & 0 & 0 & 0 & 0 \\
a_{41} & 0 & 0 & 0 & 0 \\
a_{51} & 0 & 0 & 0 & 0
\end{vmatrix}
\]

206. Prove that if a determinant of order \( n \) has zero elements at the intersection of certain \( k \) rows and \( l \) columns, \( k + l > n \), the determinant is equal to zero.

207. Solve the equation
\[
\begin{vmatrix}
1 & x & x^2 & \cdots & x^{n-1} \\
1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\
1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\
& & \ddots & \ddots & \ddots \\
1 & a_n & a_n^2 & \cdots & a_n^{n-1}
\end{vmatrix} = 0,
\]
where \( a_1, a_2, \ldots, a_n \) are distinct numbers.

208.
\[
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 1-x & 1 & \cdots & 1 \\
1 & 1+2-x & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1-n-x
\end{vmatrix} = 0.
\]

209. Find an element of a determinant of order \( n \) that is symmetric to the element \( a_{i,j} \) about the secondary diagonal.

210. Find an element of a determinant of order \( n \) that is symmetric to the element \( a_{i,k} \) about the "centre" of the determinant.

211. We say the position of an element \( a_{i,j} \) of a determinant is even or odd according as the sum of elements of its row and column is even or odd. Find the number of elements of a determinant of order \( n \) residing in even and odd positions.

212. How will an \( n \)-th order determinant change if the first column is put in the last place and the other columns are moved leftwards with all positions retained?

213. What change does a determinant of order \( n \) make if the rows are given in reversed order?

214. How does a determinant change if each element is replaced by an element symmetric to the given one about the "centre" of the determinant?

215. How does a determinant change if each element is replaced by an element symmetric to the given one about the secondary diagonal?

216. A determinant is said to be skew-symmetric if its elements symmetric about the principal diagonal differ in sign, that is, \( a_{ij} = -a_{ji} \) for any indices \( i, j \).

Prove that a skew-symmetric determinant of odd order is equal to zero.

217. Prove that a determinant whose elements are symmetric about the principal diagonal and are complex conjugate numbers (real numbers, as a special case) is a real number.

218. For what values of \( n \) will all determinants of order \( n \) whose elements satisfy the conditions
\[
(a) \ a_{jk} \text{ is real for } j > k \\
(b) \ a_{ik} = -t \ a_{jk} \text{ for } j > k \quad (t = \sqrt{-1})
\]
be real?

219. For what values of \( n \) will all determinants of order \( n \) whose elements satisfy the conditions (a) and (b) of the preceding problem be pure imaginaries?

220. Show that for odd \( n \) all determinants of order \( n \) whose elements satisfy the conditions (a) and (b) of problem 218 have the form \( a (1 \pm i) \) where \( a \) is a real number.

221. How will a determinant of order \( n \) change if the signs of all its elements are reversed?

222. How will a determinant change if each element \( a_{ij} \) is multiplied by \( c^{i-j} \), where \( c \neq 0 \)?

223. Prove that every term of a determinant involves an even number of elements occupying odd sites; and that there is an even number of elements occupying even sites if the deter
224. Prove that a determinant remains unchanged if the signs of all elements in odd positions are changed; but if the signs of all elements in even positions are changed, then the determinant remains unchanged if it is of even order, but changes sign if it is of odd order.

225. Prove that a determinant remains unchanged if to each row, with the exception of the last row, the following row is added.

226. Prove that a determinant remains unchanged if to each column (from the second onward) one adds the preceding column.

227. Prove that a determinant remains unchanged if from each row (except the last one) we subtract all succeeding rows.

228. Prove that a determinant remains unchanged if to each column (beginning with the second one) we add all preceding columns.

229. How does a determinant change if from each row (except the last one) we subtract the following row and from the last row we subtract the earlier first row?

230. What change does a determinant undergo if to each column (beginning with the second) we add the preceding column and at the same time add the last to the first?

231. What change does an nth-order determinant undergo if its matrix is turned through 90° about the "centre"?

232. What is the determinant whose sum of rows with even numbers is equal to the sum of rows with odd numbers?

233. Find the sum of all determinants of order \( n \geq 2 \), in each of which one element in each row and each column is equal to unity, the others being zero. How many such determinants are there in all?

234. Find the sum of the determinants of order \( n \geq 2 \):

\[
\begin{vmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]

where the sum is taken over all values \( \alpha_1, \alpha_2, \ldots, \alpha_n \) that can vary 1 to \( n \) independently.

235. Let all elements of the determinant

\[
\begin{vmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]

be single-valued integers. Denote by \( N_i \) the number written by the digits of the \( i \)th row of the determinant with the arrangement retained (\( a_{1n} \) is the number of units, \( a_{1n-1} \) the number of tens, and so on). Prove that the value of the determinant is divisible by the greatest common divisor of the numbers \( N_1, N_2, \ldots, N_n \).

236. Compute the following determinant by expanding in terms of the third row:

\[
\begin{vmatrix}
    2 & -3 & 4 & 1 \\
    4 & -3 & 2 & 3 \\
    b & c & d & 2 \\
    3 & -4 & 4 & 3
\end{vmatrix}
\]

237. Expanding in terms of the second column, compute the determinant

\[
\begin{vmatrix}
    5 & a & 2 & -4 \\
    4 & b & 4 & -3 \\
    2 & c & 3 & -2 \\
    4 & d & 5 & -4
\end{vmatrix}
\]

Compute the following determinants:

238. \( \begin{vmatrix} a & 3 & 0 & 5 \end{vmatrix} \)

239. \( \begin{vmatrix} 1 & 0 & 2 & a \end{vmatrix} \)

240. \( \begin{vmatrix} x & a & b & 0 & c \end{vmatrix} \)

241. Let \( M_{ij} \) be a minor of element \( a_{ij} \) of determinant \( D \).

Show that if \( D \) is a symmetric determinant or a skew-symmetric determinant of odd order, then \( M_{ij} = M_{ji} \), but if \( D \) is a skew-symmetric determinant of even order, then \( M_{ij} = -M_{ji} \).

242. Let \( D \) be a determinant of order \( n \geq 1 \). Show that if \( D \) and \( D' \) are determinants obtained from \( D \) by replacing each element \( a_{ij} \) by \( a_{ij} + k \) for some constant \( k \), then \( D' = D \).
by its cofactor $A_1$, for $D'$ and by its minor $M_{ij}$ for $D''$. The determinant $D'$ is said to be the
adjugate determinant of $D$. For expressing $D'$ in terms of $D$ see problem 50h.

243. Compute the following determinant without expanding it:

$$\begin{vmatrix}
| \frac{a}{b} & \frac{b}{c} & \frac{c}{a} \\
| \frac{b}{c} & \frac{c}{a} & \frac{a}{b} \\
| \frac{c}{a} & \frac{a}{b} & \frac{b}{c} \\
\end{vmatrix}$$

Prove the following identities without expanding the determinants.

*244. $\begin{vmatrix}
| 0 & x & y \\
| x & 0 & z \\
| y & z & 0 \\
\end{vmatrix} = \begin{vmatrix}
| 0 & 1 & 1 \\
| 1 & 0 & z^2 \\
| 1 & 1 & 0 \\
\end{vmatrix}$

*245. $\begin{vmatrix}
| a_1 & a_1^2 & \ldots & a_1^{n-2} & a_1^n \\
| a_2 & a_2^2 & \ldots & a_2^{n-2} & a_2^n \\
| \vdots & \vdots & \ddots & \vdots & \vdots \\
| a_n & a_n^2 & \ldots & a_n^{n-2} & a_n^n \\
\end{vmatrix} = (a_1 + a_2 + \ldots + a_n) \begin{vmatrix}
| 1 & a_1 & \ldots & a_1^{n-2} & a_1^n \\
| 1 & a_2 & \ldots & a_2^{n-2} & a_2^n \\
| \vdots & \vdots & \ddots & \vdots & \vdots \\
| 1 & a_n & \ldots & a_n^{n-2} & a_n^n \\
\end{vmatrix}$

*246. $\begin{vmatrix}
| a_1 & a_1 a_2 & \ldots & a_1 a_2^{n-1} & a_1 a_2^n \\
| a_2 & a_2 a_3 & \ldots & a_2 a_3^{n-1} & a_2 a_3^n \\
| \vdots & \vdots & \ddots & \vdots & \vdots \\
| a_n & a_n a_{n-1} & \ldots & a_n a_{n-1}^{n-1} & a_n a_{n-1}^n \\
\end{vmatrix} = \sum_{k_1, k_2, \ldots, k_{n-1}} a_{k_1} a_{k_2} \ldots a_{k_{n-1}} a_{k_{n-1}+1}$

where the sum is taken over all combinations of $n$ numbers $1, 2, 3, \ldots, n$ with respect to $a_1, a_2, a_3, \ldots, a_n$.

Using the properties of determinant including expansions in terms of a row or column, prove the following identities:

*247. $\begin{vmatrix}
| \cos \frac{x}{2} & \sin \frac{x}{2} & \cos \frac{y}{2} \\
| \cos \frac{y}{2} & \cos \frac{x}{2} & \sin \frac{z}{2} \\
| \cos \frac{z}{2} & \cos \frac{y}{2} & \cos \frac{x}{2} \\
\end{vmatrix} = \frac{1}{2} [\sin (\alpha - \beta) + \sin (\gamma - \beta)] \sin (\alpha - \beta) = \frac{1}{2} [\sin (\beta - \alpha) + \sin (\gamma - \beta)] \sin (\alpha - \beta)$

248. $\begin{vmatrix}
| \cos \frac{x}{2} & \sin \frac{x}{2} & \cos \frac{y}{2} \\
| \sin \frac{x}{2} & \cos \frac{x}{2} & \sin \frac{y}{2} \\
| \sin \frac{y}{2} & \sin \frac{y}{2} & \cos \frac{y}{2} \\
\end{vmatrix} = \sin (\alpha - \beta) \cos \alpha \cos \beta + \sin (\beta - \gamma) \cos \beta \cos \gamma$

249. $\begin{vmatrix}
| a & b & c \\
| b & c & a \\
| c & a & b \\
\end{vmatrix} = 2abc (a + b + c)^3$.

250. $\begin{vmatrix}
| 1 & 1 & 1 \\
| a & \frac{1}{a} & \frac{1}{b} \\
| \frac{1}{b} & \frac{1}{b} & \frac{1}{c} \\
\end{vmatrix} = \frac{1}{(a+b)(a+c)(b+c)(a+b+c)}$

251. $\begin{vmatrix}
| 1 & \frac{1}{a} & \frac{1}{a+x} \\
| a+x & a+y & a+z \\
| a+y & a+z & a+x \\
\end{vmatrix} = \frac{1}{(a+b)(a+c)(b+c)(a+b+c)}$
| \( a^2 + (1 - a^2) \cos \varphi \) | \( ab (1 - \cos \varphi) \) | \( ac (1 - \cos \varphi) \) \\ 
| \( ba (1 - \cos \varphi) \) | \( b^2 + (1 - b^2) \cos \varphi \) | \( bc (1 - \cos \varphi) \) \\ 
| \( ca (1 - \cos \varphi) \) | \( cb (1 - \cos \varphi) \) | \( c^2 + (1 - c^2) \cos \varphi \) 

when \( a^2 + b^2 + c^2 = 1 \).

See 4. Evaluating Determinants with Numerical Elements

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**269.**

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**270.**

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\[ \text{det} = \cos^2 \theta, \]

\[ \text{det} = \cos \theta \sin \theta, \]

\[ \text{det} = \cos \theta. \]
Sec. 5. Methods of Computing Determinants of the nth Order

Introduction. The approach to computing determinants with numerical elements, which consist in making elements of a certain row (or column) except one vanish in a subsequent reduction in the order, becomes extremely unwieldy in the case of determinants of the given order with literal elements. In the general case, this leads to an expression like that obtained when computing a determinant by direct application of the definition of a determinant. The method is all the more inconvenient when dealing with determinants of arbitrary order with literal or numerical elements.

There is no general method for computing such determinants (with the exception of the expression given in the definition of a determinant). In handling special types of determinants, one makes use of a variety of methods of computation that lead to simpler expressions of the determinant (that is, such that contain a smaller number of operations) than that obtained from the definition. We will now investigate several of the most widely used methods and will then offer problems in each method and also problems in which the student himself chooses the most suitable method of solution. To simplify orientation, problems involving the Laplace theorem and multiplication of determinants are collected in separate sections.

1. Method of reducing to triangular form. This method consists in transforming the determinant to a form in which all elements to one side of a diagonal are equal to zero. The case of the secondary diagonal can be reduced to the case of the principal diagonal by reversing the order of the row (or columns). The resulting determinant is equal to the product of the elements of the principal diagonal.

3-0281
Example 1. Compute the following determinant of order 3:

\[
D = \begin{vmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{vmatrix}
\]

add all columns to the first one:

\[
D = \begin{vmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1 \\
\end{vmatrix} = (-1)^{n-1}.
\]

Example 2. Compute the determinant

\[
D = \begin{vmatrix}
x & x & x \\
x & x & x \\
x & x & x \\
\end{vmatrix}
\]

\[
D = x^3 - 3x^2 + 2x
\]

By comparing the separate terms of the determinant with the terms of the product of linear factors, one finds the product of the determinant by this product and the quotient obtained from the division of the determinant by this product and the determinant.
The product contains the term \( z^1 \) with the coefficient \( -1 \), and the determinant itself contains the same term \( z^1 \) with the coefficient \( +1 \). Thus

\[
D = -x^3 + 3x^2 - 2x + 6.
\]

Example 3. The method of isolating linear factors. Consider the underdetermined determinant of the form

\[
\begin{vmatrix}
1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\
1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \ldots & x_n^{n-1}
\end{vmatrix}
\]

The method of isolating linear factors consists in transforming the given determinant into terms of a row or a column of the determinant of the same type but one less factor. The result is called a recurrence relation.

Thus

\[
D_n = (x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1}).
\]

3. The method of recurrence relations. It consists in transforming the given determinant in terms of a row or a column of the determinants of the same type but one less factor. Then, using the general form of the determinant, compute directly many determinants in which there were in the right-hand member of the recurrence relation. The higher-order determinants are computed successively from the recurrence relation. If one has an expression for a determinant of any order, then computing from the recurrence relation several lower order determinants, one attempts to obtain the general form of the desired expression and then proves the validity of the expression for arbitrary \( n \) with the aid of the recurrence relation and by induction on \( n \).

The first recurrence relation that expresses the determinant the expression of the determinant from the same recurrence relation will be of the form

\[
D_n = (x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1}).
\]

and so on until the form

\[
D_n = (x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1}).
\]

for the
$p, q$ are constants, that is, quantities independent of $n$. 

$D_n$ is computed as a term of a geometric progression of the form $p^n - q^n$. Here $D_1$ is a first-order determinant of the type, that is to say, an element of the determinant lying in the upper left-hand corner.

Let $a$ and $b$ be the roots of the quadratic equation $a^2 - q = 0$. Then $p = a + b$. If $q = -ab$ and equation (4) can be rewritten thus:

$$D_n = \beta D_{n-1} = \alpha (D_{n-1} - \beta D_{n-2})$$  \hspace{1cm} (2)

$$D_n - \alpha D_{n-1} = \beta (D_{n-1} - \alpha D_{n-2}).$$  \hspace{1cm} (3)

Next consider that $\alpha \neq \beta$.

From the formula for the $(n - 1)$th term of the geometric progression, we derive, from (2) and (3),

$$D_n = \alpha D_{n-1} + \beta D_{n-2}$$

and

$$D_n - \alpha D_{n-1} = \beta D_{n-2} (D_2 - \alpha D_1)$$

where

$$C = \frac{D_2 - \alpha D_1}{\alpha - \beta}, \quad C_2 = \frac{D_2}{\alpha}, \quad C_3 = \frac{D_2}{\alpha}.$$

In the first determinant, subtract the last column from all the others, and then expand the second determinant in terms of the last column:

$$D_n = x_{a_1} (a_2 - x) \cdots (a_n - x) + (a_n - x) D_{n-1}$$

This is a recurrence relation. Putting into it an analogous expression for $D_{n-1}$, we get

$$D_n = x_{a_1 = x} (a_2 - x) \cdots (a_n - x) + x (a_1 - x) (a_2 - x) \cdots (a_{n-1} - x) (a_n - x) + D_{n-1} (a_n - x).$$

Repeating this reasoning $n - 2$ times and remembering that $D_1 = a_1 - x + (a_1 - x)$, we obtain

$$D_n = x_{a_1 = x} (a_2 - x) \cdots (a_n - x) + x (a_1 - x) (a_2 - x) \cdots (a_{n-2} - x) (a_{n-1} - x) \cdots + D_{n-1} (a_n - x).$$

which coincides with the result of the author by L.Ya. Okunev.
Example 6. Calculate the nth-order determinant
\[
\begin{vmatrix}
5 & 3 & 0 & \ldots & 0 \\
5 & 3 & 0 & \ldots & 0 \\
0 & 2 & 5 & 3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 2 & 5
\end{vmatrix}
\]

Expanding by the first row, we find the recurrence relation
\[
D_n = 5D_{n-1} + 10D_{n-2}.
\]

The equation \( D_n = \alpha a_n \) yields the roots
\[
\lambda_1 = 5, \quad \lambda_2 = -1.
\]

By formula \( D_n = \alpha a_n \), \( \alpha = 3^{n+1} - 2^{n+1} \).

5. The method of changing the elements of a determinant. This method is applicable when we want to change a determinant by the same number with the cofactors of all elements in a row. The method is based on the following property: when a number \( x \) is added to all elements in a row, the determinant is increased by the product of the cofactors of all elements in that row. Let us prove this property.

Decompose \( D' \) into two determinants with respect to the first row, then each of them into two determinants with respect to the second row and so on.

The terms containing more than one row of elements are equal to zero, since the terms containing one row of elements are equal to zero.

Thus, expanding the determinant \( D' \) in terms of that row to get \( D' = D + x \).

Example 7. Compute the determinant
\[
\begin{vmatrix}
a_1 & a_2 & \ldots & a_n \\
0 & a_{n-1} & \ldots & a_1 \\
0 & 0 & \ldots & a_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_1
\end{vmatrix}
\]

We can add to the first row, this determinant can be expanded into two determinants, each of which can be expanded into two determinants with respect to the second row, and so on. When we reach the last row, we have 2^n determinants with respect to the last row.

Example 8. Compute the determinant \( D_4 \) of example 7.

Subtracting \( x \) from all its elements, we obtain the determinant
\[
\begin{vmatrix}
a_1 - x & 0 & \ldots & 0 \\
0 & a_{n-1} - x & \ldots & 0 \\
0 & 0 & \ldots & a_{n-2} - x \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_1 - x
\end{vmatrix}
\]

The cofactors of the elements of \( D \) on the principal diagonal are equal to zero, and the cofactor of each element on the principal diagonal is equal to the product of the remaining elements \( \beta, \) and the principal diagonal. Therefore
\[
D_n = (a_1 - x) (a_2 - x) \ldots (a_n - x)
\]

The cofactor of each element of \( D \) on the principal diagonal is equal to zero, and the cofactor of each element on the principal diagonal is equal to the product of the remaining elements. Thus,
\[
D_n = (a_1 - x) (a_2 - x) \ldots (a_n - x)
\]

This is what we sought.
Evaluate the following determinants by reduction to triangular form (wherever the order of the determinant cannot be judged exactly, it is assumed to be of order $n$):

279. \[ \begin{vmatrix} 1 & 2 & 3 & \ldots & n \\ -1 & 0 & 3 & \ldots & n \\ -1 & -2 & 0 & \ldots & n \\ -1 & -2 & -3 & \ldots & 0 \end{vmatrix} \]

280. \[ \begin{vmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ 2 & 3 & 4 & \ldots & n & n \\ 3 & 4 & 5 & \ldots & n & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & n & \ldots & n & n \end{vmatrix} \]

281. \[ \begin{vmatrix} x_1 & a_{12} & a_{13} & \ldots & a_{1n} \\ x_1 & x_2 & a_{23} & \ldots & a_{2n} \\ x_1 & x_2 & x_3 & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \ldots & x_n \end{vmatrix} \]

282. \[ \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ a_1 & a_2 & a_3 & \ldots & a_n \\ a_1 & a_2 & a_3 & \ldots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_n \end{vmatrix} \]

283. \[ \begin{vmatrix} 1 & 2 & 3 & \ldots & 2 \\ 2 & 3 & 2 & \ldots & 2 \\ 2 & 2 & 3 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \ldots & 2 \end{vmatrix} \]

284. \[ \begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_1 & a_2 & a_3 & \ldots & a_n \\ a_1 & a_2 & a_3 & \ldots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_n \end{vmatrix} \]

286. Compute a determinant of order $n$ whose elements are given by the conditions $a_{ij} = \min\{i, j\}$.

287. Compute a determinant of order $n$ whose elements are given by $a_{ij} = \max\{i, j\}$.

*288. Compute a determinant of order $n$ whose elements are given by $a_{ij} = |i - j|$.

Evaluate the following determinants using the method of isolation of linear factors:

289. \[ \begin{vmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & x+1 & 3 & \ldots & n \\ 1 & 2 & x+1 & \ldots & n \\ 1 & 2 & 3 & \ldots & x+1 \end{vmatrix} \]

290. \[ \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 3 & \ldots & 1 \\ 1 & 1 & 3 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 3 & \ldots & 1 \end{vmatrix} \]

291. \[ \begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_1 & x & a_3 & \ldots & a_n \\ a_1 & a_3 & x & \ldots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_3 & a_5 & \ldots & x \end{vmatrix} \]

292. \[ \begin{vmatrix} -x & a & b & c \\ a & -x & c & b \\ b & c & -x & a \\ c & b & a & -x \end{vmatrix} \]

293. \[ \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & x & 2 \\ 1 & 2 & 3 & x \\ 1 & 2 & 3 & 1 \end{vmatrix} \]

294. \[ \begin{vmatrix} 1 & 1 & 4 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \\ 1 & 1 & 1 & 1 \end{vmatrix} \]
Compute the following determinants by the method of recurrence relations:

295. \[
\begin{vmatrix}
  a_1 & a_2 & a_3 & \ldots & a_n \\
  a_2 & a_3 & a_4 & \ldots & a_{n+1} \\
  a_3 & a_4 & a_5 & \ldots & a_{n+2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_n & a_{n+1} & a_{n+2} & \ldots & a_{2n-1} \\
\end{vmatrix}
\]

296. \[
\begin{vmatrix}
  a_0 & a_1 & a_2 & \ldots & a_n \\
  -y_1 & x_1 & 0 & \ldots & 0 \\
  0 & y_2 & x_2 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & x_n \\
\end{vmatrix}
\]

297. \[
\begin{vmatrix}
  0 & 1 & 0 & \ldots & 0 \\
  1 & a_1 & 0 & \ldots & 0 \\
  1 & 0 & a_2 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & 0 & 0 & \ldots & a_n \\
\end{vmatrix}
\]

298. \[
\begin{vmatrix}
  1 & 0 & 0 & \ldots & 1 \\
  1 & a_1 & 0 & \ldots & 0 \\
  1 & 0 & a_2 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & 0 & 0 & \ldots & 1 \\
\end{vmatrix}
\]

299. \[
\begin{vmatrix}
  2 & 1 & 0 & \ldots & 0 \\
  1 & 2 & 1 & \ldots & 0 \\
  0 & 1 & 2 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 2 \\
\end{vmatrix}
\]

300. \[
\begin{vmatrix}
  3 & 2 & 0 & \ldots & 0 \\
  1 & 3 & 2 & \ldots & 0 \\
  0 & 1 & 3 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 3 \\
\end{vmatrix}
\]

301. \[
\begin{vmatrix}
  7 & 5 & 0 & \ldots & 0 \\
  2 & 7 & 5 & \ldots & 0 \\
  0 & 2 & 7 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 7 \\
\end{vmatrix}
\]

302. \[
\begin{vmatrix}
  5 & 6 & 0 & \ldots & 0 \\
  5 & 6 & 0 & \ldots & 0 \\
  4 & 5 & 2 & \ldots & 0 \\
  0 & 1 & 3 & 2 & \ldots & 0 \\
  0 & 0 & 1 & 3 & 2 & \ldots & 0 \\
\end{vmatrix}
\]

303. \[
\begin{vmatrix}
  1 & 2 & 0 & \ldots & 0 \\
  3 & 4 & 3 & 0 & \ldots & 0 \\
  0 & 2 & 5 & 3 & \ldots & 0 \\
  0 & 0 & 2 & 5 & 3 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 & \ldots & 5 & 3 \\
  0 & 0 & 0 & 0 & \ldots & 2 & 5 \\
\end{vmatrix}
\]

304. \[
\begin{vmatrix}
  \alpha & \beta & 0 & \ldots & 0 \\
  0 & 1 & \alpha & \beta & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 1 & \alpha & \beta \\
\end{vmatrix}
\]

Compute the following determinants by representing them as a sum of determinants:

305. \[
\begin{vmatrix}
  x & -a_1 & a_2 & a_3 & \ldots & a_n \\
  a_1 & x & -a_2 & a_3 & \ldots & a_n \\
  a_2 & a_1 & x & -a_3 & \ldots & a_n \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{n-1} & a_{n-2} & a_{n-3} & \ldots & x & -a_n \\
\end{vmatrix}
\]

306. \[
\begin{vmatrix}
  x_1 & a_2 & \ldots & a_n \\
  a_1 & x_2 & \ldots & a_n \\
  a_2 & a_1 & \ldots & x_n \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n-1} & a_{n-2} & \ldots & x \\
\end{vmatrix}
\]

307. \[
\begin{vmatrix}
  0 & 1 & 1 & 1 & \ldots & 1 \\
  x_1 & a_1 & 0 & 0 & \ldots & 0 \\
  x_2 & x_1 & a_2 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_n & x_{n-1} & x_{n-2} & \ldots & a_3 & a_2 & a_1 \\
\end{vmatrix}
\]

308. \[
\begin{vmatrix}
  x_1 & a_1b_2 & a_1b_3 & \ldots & a_1b_n \\
  a_2b_1 & x_2 & a_2b_3 & \ldots & a_2b_n \\
  a_3b_1 & a_3b_2 & x_3 & \ldots & a_3b_n \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_nb_1 & a_nb_2 & a_nb_3 & \ldots & x_n \\
\end{vmatrix}
\]
Compute the following determinants (the order of any determinant that is not clear is assumed to be $n$):

309. $\begin{vmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ 1 & 2 & 3 & \ldots & n-1 & n \\ 1 & 2 & \ldots & 2n-3 & n \\ 1 & 2 & \ldots & n-1 & 2n-1 \end{vmatrix}$

310. $\begin{vmatrix} 1 & a_1 & a_2 & \ldots & a_n & a_n \\ 1 & a_1 + b_1 & a_2 & \ldots & a_n & a_n \\ 1 & a_1 & a_2 + b_2 & \ldots & a_n & a_n \\ 1 & a_1 & a_2 & \ldots & a_n + b_n & a_n \end{vmatrix}$

311. $\begin{vmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 3 \\ 2 & 2 & \ldots & 2 & 2 \\ 2 & 2 & \ldots & 2 & 2 \end{vmatrix}$

312. $\begin{vmatrix} 1 & n & n & \ldots & n \\ n & n & n & \ldots & n \\ 2 & 2 & 2 & \ldots & 2 \\ 2 & 2 & 2 & \ldots & 2 \\ 2 & 2 & \ldots & 2 & 2 \end{vmatrix}$

313. $\begin{vmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & 0 & x \end{vmatrix}$

314. $\begin{vmatrix} 1 & n & 1 & 1 & 1 \\ 1 & 1 & -n & 1 & 1 \\ 1 & 1 & 1 & -n & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix}$

315. $\begin{vmatrix} 1 & 1 & 1 & -n \\ 1 & 1 & \ldots & -n \\ 1 & \ldots & \ldots & \ldots & \ldots \\ -n & \ldots & 1 & 1 & 1 \end{vmatrix}$

316. $\begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & 1 \end{vmatrix}$

317. $\begin{vmatrix} n & 1 & 1 & \ldots & 1 \\ 1 & n & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \end{vmatrix}$

318. $\begin{vmatrix} a & b & \ldots & b & b \\ b & a & \ldots & b & b \\ b & b & \ldots & a & b \\ b & b & b & a & b \end{vmatrix}$

319. $\begin{vmatrix} 1 & 2 & 0 & 0 & \ldots & 0 \\ 1 & 3 & 2 & 0 & \ldots & 0 \\ 0 & 1 & 3 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 3 \end{vmatrix}$

320. $\begin{vmatrix} 1 & b_1 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 1 & -b_2 & b_3 & \ldots & 0 & 0 \\ 0 & -1 & 1 & -b_3 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 1 \end{vmatrix}$

321. $\begin{vmatrix} x & a_1 & a_2 & \ldots & a_{n-1} \\ a_1 & x & a_2 & \ldots & a_{n-1} \\ a_1 & a_2 & x & \ldots & a_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & a_2 & a_3 & \ldots & x \end{vmatrix}$

322. $\begin{vmatrix} a_0 & -4 & 0 & 0 & \ldots & 0 & 0 \\ a_1 & 1 & -4 & 0 & \ldots & 0 & 0 \\ a_2 & 0 & 1 & -4 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1} & 0 & 0 & 0 & \ldots & 1 & 1 \\ a_n & 0 & 0 & 0 & \ldots & 1 & 1 \end{vmatrix}$
| 323 | \( \begin{bmatrix} 1 & 2 & \ldots & n-1 & n \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \) |

| 324 | \( \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ a & 1 & \ldots & 0 & 0 \\ a^2 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^n & 0 & \ldots & 0 & 1 \end{bmatrix} \) |

| 325 | \( \begin{bmatrix} 1 & a & \ldots & a^{n-2} & a^{n-1} \\ a & a^2 & \ldots & a^{n-3} & a^{n-2} \\ a^2 & a^3 & \ldots & a^{n-4} & a^{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{n-1} & a^n & \ldots & a & 1 \end{bmatrix} \) |

| 326 | \( \begin{bmatrix} 1 & a^2 & \ldots & a^{n-1} \\ a & a^2 & \ldots & a^{n-2} \\ a^2 & a^3 & \ldots & a^{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & a^n & \ldots & a^2 \\ a^{n-1} & a^n & \ldots & a^2 \\ a^{n-1} & a^n & \ldots & a^2 \\ \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & a^n & \ldots & a^2 \\ a^{n-1} & a^n & \ldots & a^2 \\ a^{n-1} & a^n & \ldots & a^2 \end{bmatrix} \) |

| 327 | \( \begin{bmatrix} 1 & 1 & \ldots & 1 \\ a & a^2 & \ldots & a^n \\ a^2 & a^3 & \ldots & a^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a^n & a^{n+1} & \ldots & a^{2n-1} \end{bmatrix} \) |

| 328 | \( \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \) |

| 329 | \( \begin{bmatrix} 1 & a^2 & \ldots & a^{n-1} \\ a & a^2 & \ldots & a^{n-2} \\ a^2 & a^3 & \ldots & a^{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & a^n & \ldots & a^2 \\ a^{n-1} & a^n & \ldots & a^2 \\ a^{n-1} & a^n & \ldots & a^2 \\ \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & a^n & \ldots & a^2 \\ a^{n-1} & a^n & \ldots & a^2 \\ a^{n-1} & a^n & \ldots & a^2 \end{bmatrix} \) |

| 330 | \( \begin{bmatrix} 1 & x_n & x_{n+1} \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \) |

| 331 | \( \begin{bmatrix} 1 & x_n & x_{n+1} \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \) |

| 332 | \( \begin{bmatrix} 1 & x_n & x_{n+1} \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \) |

| 333 | \( \begin{bmatrix} 1 & x_n & x_{n+1} \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \) |

| 334 | \( \begin{bmatrix} 1 & x_n & x_{n+1} \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \) |
where $\psi_n(x) = x^k + a_{k-1} x^{k-1} + a_{k-2} x^{k-2} + \ldots + a_0$.

335. \[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
f_1(\cos \varphi_1) & f_1(\cos \varphi_2) & \ldots & f_1(\cos \varphi_n) \\
f_2(\cos \varphi_1) & f_2(\cos \varphi_2) & \ldots & f_2(\cos \varphi_n) \\
\vdots & \vdots & \ddots & \vdots \\
f_{n-1}(\cos \varphi_1) & f_{n-1}(\cos \varphi_2) & \ldots & f_{n-1}(\cos \varphi_n)
\end{bmatrix}
\]

where $f_k(x) = a_k x^k + a_{k-1} x^{k-1} + a_{k-2} x^{k-2} + \ldots + a_0$.

336. \[
\begin{bmatrix}
1 & C_2^1 & \ldots & C_{2n-1}^{1} \\
1 & C_2^2 & \ldots & C_{2n-1}^{2} \\
1 & C_2^3 & \ldots & C_{2n-1}^{3} \\
1 & C_2^{n} & \ldots & C_{2n-1}^{n}
\end{bmatrix}
\]

where $C_k^i = \frac{x(x-1)(x-2) \ldots (x-k+1)}{k!}$.

337. \[
\begin{bmatrix}
(2n-1)^n & (2n-2)^n & \ldots & n^n & (2n)^n \\
(2n-1)^{n-1} & (2n-2)^{n-1} & \ldots & n^{n-1} & (2n)^{n-1} \\
2n-1 & 2n-2 & \ldots & n & 2n
\end{bmatrix}
\]

338. \[
\begin{bmatrix}
x_1 & x_2 & \ldots & x_n \\
x_1^2 & x_2^2 & \ldots & x_n^2 \\
x_1^3 & x_2^3 & \ldots & x_n^3 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1}
\end{bmatrix}
\]

339. \[
\begin{bmatrix}
1 & 2 & 3 & \ldots & n \\
1 & 2^2 & 3^2 & \ldots & n^2 \\
1 & 2^3 & 3^3 & \ldots & n^3 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2^{n-1} & 3^{n-1} & \ldots & n^{n-1}
\end{bmatrix}
\]

340. \[
\begin{bmatrix}
a_1 & a_1^{-1} b_1 & a_1^{-2} b_1^2 & \ldots & b_1^n \\
a_2 & a_2^{-1} b_2 & a_2^{-2} b_2^2 & \ldots & b_2^n \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

341. \[
\begin{bmatrix}
\sin^{r-1} \alpha_1 \sin^{r-2} \alpha_1 \cos \alpha_1 & \ldots & \cos^{r-1} \alpha_1 \\
\sin^{r-1} \alpha_2 \sin^{r-2} \alpha_2 \cos \alpha_2 & \ldots & \cos^{r-1} \alpha_2 \\
\vdots & \vdots & \ddots & \vdots \\
\sin^{r-1} \alpha_n \sin^{r-2} \alpha_n \cos \alpha_n & \ldots & \cos^{r-1} \alpha_n
\end{bmatrix}
\]

342. \[
\begin{bmatrix}
f_1(x_1, y_1) & y_1 f_{n-1}(x_1, y_1) & \ldots & y_1^{n-1} f_1(x_1, y_1) & y_1^n \\
f_1(x_2, y_2) & y_2 f_{n-1}(x_2, y_2) & \ldots & y_2^{n-1} f_1(x_2, y_2) & y_2^n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
f_n(x_{n+1}, y_{n+1}) & y_{n+1} f_{n-1}(x_{n+1}, y_{n+1}) & \ldots & y_{n+1}^{n-1} f_1(x_{n+1}, y_{n+1}) & y_{n+1}^n
\end{bmatrix}
\]

where $f_i(x, y)$ is a homogeneous polynomial in $x, y$ of degree $i$.

*343. \[
\begin{bmatrix}
a_1 & x_1 & x_1^2 & \ldots & x_1^{i-1} \\
a_2 & x_2 & x_2^2 & \ldots & x_2^{i-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_n & x_n & x_n^2 & \ldots & x_n^{i-1}
\end{bmatrix}
\]

*344. \[
\begin{bmatrix}
x_1 & x_1^2 & x_1^3 & \ldots & x_1^{i-2} x_1^i \\
x_2 & x_2^2 & x_2^3 & \ldots & x_2^{i-2} x_2^i \\
\vdots & \vdots & \ddots & \vdots \\
x_n & x_n^2 & x_n^3 & \ldots & x_n^{i-2} x_n^i
\end{bmatrix}
\]

*345. \[
\begin{bmatrix}
x_1^2 & x_1^3 & x_1^n \\
x_2^2 & x_2^3 & x_2^n \\
\vdots & \vdots & \ddots \\
x_n^2 & x_n^3 & x_n^n
\end{bmatrix}
\]

*346. \[
\begin{bmatrix}
x_1 & x_1^2 & x_1^3 & \ldots & x_1^{i-1} x_1^i \\
x_2 & x_2^2 & x_2^3 & \ldots & x_2^{i-1} x_2^i \\
\vdots & \vdots & \ddots & \vdots \\
x_n & x_n^2 & x_n^3 & \ldots & x_n^{i-1} x_n^i
\end{bmatrix}
\]

*347. \[
\begin{bmatrix}
x_1 (x_1 - 1) & x_1^2 (x_1 - 1) & \ldots & x_1^{i-1} (x_1 - 1) \\
x_2 (x_2 - 1) & x_2^2 (x_2 - 1) & \ldots & x_2^{i-1} (x_2 - 1) \\
\vdots & \vdots & \ddots & \vdots \\
x_n (x_n - 1) & x_n^2 (x_n - 1) & \ldots & x_n^{i-1} (x_n - 1)
\end{bmatrix}
\]

\*
\[ \begin{vmatrix} 1 + x_1 & 1 + x_2 & \cdots & 1 + x_n \\ 1 + x_2 & 1 + x_3 & \cdots & 1 + x_n \\ \vdots & \vdots & \ddots & \vdots \\ 1 + x_n & 1 + x_1 & \cdots & 1 + x_{n-1} \end{vmatrix} \]

**358.**

\[ \begin{vmatrix} \cos \varphi_1 & \cos 2\varphi_1 & \cdots & \cos (n-1) \varphi_1 \\ \cos \varphi_2 & \cos 2\varphi_2 & \cdots & \cos (n-1) \varphi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \cos \varphi_n & \cos 2\varphi_n & \cdots & \cos (n-1) \varphi_n \end{vmatrix} \]

**359.**

\[ x_{1}y_{1} + a_{1}b_{1} + a_{2}b_{2} + \cdots + a_{n}b_{n} \]

The order of the determinants is equal to 2n.

**360.**

\[ a_{1}b_{1} + a_{2}b_{2} + a_{3}b_{3} + \cdots + a_{n}b_{n} \]

\[ x_{1} + a_{1}b_{1} + a_{2}b_{2} + a_{3}b_{3} + \cdots + a_{n}b_{n} \]

The order of the determinant is equal to 2n.

**361.**

\[ a \, 0 \, \ldots \, 0 \, b \\
0 \, 0 \, \ldots \, b \\
\vdots \, \vdots \, \vdots \, \vdots \\
0 \, b \, \ldots \, a_{n} \, b_{n} \]

\[ \begin{vmatrix} x_{1} & a_{1} & a_{2} & \cdots & a_{n} \\ a_{1} & x_{2} & a_{3} & \cdots & a_{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n} & a_{n-1} & a_{n-2} & \cdots & x_{2} \\ 0 & 0 & 0 & \cdots & x_{1} \end{vmatrix} \]
A Fibonacci sequence (Fibonacci was an Italian mathematician of the 13th century) is a number sequence which begins with the numbers 1, 2 and in which each succeeding number is equal to the sum of the two preceding ones: we have the sequence 1, 2, 3, 5, 8, 13, 21, ... .

Prove that the nth term of the Fibonacci sequence is equal to the nth-order determinant:

\[
\begin{vmatrix}
1 & 1 & 0 & \ldots & 0 \\
1 & 1 & 1 & \ldots & 0 \\
0 & 1 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{vmatrix}
\]

Evaluate the following determinants:

\[
\begin{vmatrix}
9 & 5 & 0 & \ldots & 0 \\
9 & 5 & 0 & \ldots & 0 \\
0 & 9 & 5 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 49
\end{vmatrix}
\]

Using this result and the result of problem 369, obtain the expression \( \cos n \alpha \) in terms of \( \cos \alpha \).

Prove the equality

\[
\sin n \alpha = \frac{2 \cos \alpha}{\sin \alpha} \begin{vmatrix}
2 \cos \alpha & 1 & 0 & \ldots & 0 \\
1 & 2 \cos \alpha & 1 & \ldots & 0 \\
0 & 1 & 2 \cos \alpha & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 2 \cos \alpha
\end{vmatrix}
\]

where the determinant is of order \( n - 1 \) in the equality and the result of problem 369, represents the product of \( \cos \alpha \) into a polynomial of \( \cos \alpha \).
*373. Prove the following equality without computing the determinants.

\[
\begin{vmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & a_1 & a_2 & \cdots & 0 \\
1 & a_2 & a_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{n-2} & a_{n-3} & \cdots & a_n
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{vmatrix}
\]

Compute the following determinants:

*374. 
\[
\begin{vmatrix}
x & 0 & 0 & \cdots & 0 \\
x & 1 & x & 0 & \cdots & 0 \\
x & 1 & 1+x^2 & x & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x & 1 & 1+x^2 & x & \cdots & 0 \\
0 & 0 & 0 & \cdots & x & 1+x^2
\end{vmatrix}
\]

*375. 
\[
\begin{vmatrix}
1 & 2 & 3 & \cdots & n-1 & n \\
2 & 3 & 4 & \cdots & n & 1 \\
3 & 4 & 5 & \cdots & 1 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n & 1 & 2 & \cdots & n-2 & n-1
\end{vmatrix}
\]

*376. 
\[
\begin{vmatrix}
a & a+x & a+2x & \cdots & a+(n-2)x & a+(n-1)x \\
a+(n-1)x & a & a+x & \cdots & a+(n-3)x & a+(n-2)x \\
a+(n-2)x & a+(n-1)x & a & \cdots & a+(n-4)x & a+(n-3)x \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a & a+2x & a+3x & \cdots & a+(n-1)x & a
\end{vmatrix}
\]

377. 
\[
\begin{vmatrix}
1 & x^2 & x^3 & \cdots & x^{n-2} & x^{n-1} \\
1 & x & x^2 & \cdots & x^{n-3} & x^{n-2} \\
1 & x & x^2 & \cdots & x^{n-4} & x^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x & x^2 & x^3 & \cdots & x^{n-1} & 1
\end{vmatrix}
\]

378. Determine without computing the determinant how the two circulants are related.

\[
\begin{vmatrix}
a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\
a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
a_{n-1} & a_n & a_1 & \cdots & a_{n-3} & a_{n-2} & a_{n-1}
\end{vmatrix}
= \begin{vmatrix}
a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\
a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
a_{n-1} & a_n & a_1 & \cdots & a_{n-3} & a_{n-2} & a_{n-1}
\end{vmatrix}
\]

The circulants are built up of the same numbers \(a_1, a_2, \ldots, a_n\) by means of circular permutations in two opposite directions.

Compute the following determinants:

*379. 
\[
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
2 & 3 & 4 & \cdots & (n-1) \\
(2) & (3) & (4) & \cdots & (n-1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(2) & (3) & (4) & \cdots & (n-1) \\
\frac{n}{n-1} & \frac{n+1}{n-1} & \frac{n+2}{n-1} & \cdots & \frac{2n-2}{n-1}
\end{vmatrix}
\]

where \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\).

*380. 
\[
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
\frac{m}{1} & \frac{m+1}{1} & \frac{m+2}{1} & \cdots & \frac{m+n}{1} \\
\frac{m+1}{2} & \frac{m+2}{2} & \frac{m+3}{2} & \cdots & \frac{m+n+1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{m+n-1}{n} & \frac{m+n}{n} & \frac{m+n}{n} & \cdots & \frac{m+n}{n}
\end{vmatrix}
\]
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<tbody>
<tr>
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<tr>
<td></td>
<td>(2)</td>
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<tr>
<td></td>
<td>(3)</td>
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<tr>
<td></td>
<td>(n)</td>
</tr>
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<td>0 0 0 0</td>
</tr>
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<td>0 0 3</td>
</tr>
<tr>
<td>3</td>
<td>0 3 3</td>
</tr>
<tr>
<td>4</td>
<td>0 (n-1)</td>
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<tbody>
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<td>1 !</td>
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<tr>
<td></td>
<td>2 !</td>
</tr>
<tr>
<td></td>
<td>3 !</td>
</tr>
<tr>
<td></td>
<td>4 !</td>
</tr>
<tr>
<td>1 n</td>
<td>(n-1) (n-2) (n-3)</td>
</tr>
</tbody>
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<tr>
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</thead>
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<tr>
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<tr>
<td></td>
<td>(3)</td>
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<tr>
<td></td>
<td>(n)</td>
</tr>
<tr>
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<td>0 0 0 0</td>
</tr>
<tr>
<td>2</td>
<td>0 0 3</td>
</tr>
<tr>
<td>3</td>
<td>0 3 3</td>
</tr>
<tr>
<td>4</td>
<td>0 (n-1)</td>
</tr>
</tbody>
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<td>(2)</td>
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<td></td>
<td>(3)</td>
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<td>(n)</td>
</tr>
<tr>
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<td>x x x x</td>
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<td>y y y y</td>
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<td>3</td>
<td>y y y y</td>
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<td>y y y y</td>
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<table>
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<tbody>
<tr>
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<td>y a x x</td>
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<td>y y y y</td>
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<th>393.</th>
<th>a1 2 a2 x</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>y y y y</td>
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<td>y y y y</td>
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<tr>
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<td>y y y</td>
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<td>y y y</td>
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<table>
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<th>395.</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>y y y y</td>
</tr>
<tr>
<td></td>
<td>y y y y</td>
</tr>
</tbody>
</table>

| 396. | a + 1 a 0 0 0 |
|      | 1 a + 1 a 0 |
|      | 0 1 a 1 a |
|      | 0 0 0 1 a |

| 397. | n(n-1) a |
|      | -n -2 a  |
|      | 0 -a |

| 398. | cos x a 1 |
|      | 1 2 cos x |
|      | 1 0 0 0 0 |

| 399. | x x x x x |
|      | y y y y y |
|      | n-1 x x  |
|      | 0 n-2    |
|      | 0 0 n    |
|      | 0 0 0    |
that is, an expression in the form of a polynomial in
\(a_2, \ldots, a_n\). Write out in expanded form continuants
of order 4, 5 and 6.

Sec. 6. Minors, Cofactors
and the Laplace Theorem

421. How many minors of order \(k\) does an \(n\)th-order determinate have?

422. Prove that when determining the sign of a cofactor, one can use the sum of the numbers of the rows and columns of the complementary minor instead of the given minor. In other words, if \(M\) is the given minor, \(M'\) is the complementary minor, \(A\) is the cofactor of \(M\), and \(A'\) is the cofactor of \(M'\), then from \(A = eM'\), where \(e = \pm 1\), it follows that \(A' = eM\).

423. Show that the Laplace expansion of an \(n\)th-order determinant in terms of any \(k\) rows (or columns) coincides with its expansion in terms of the remaining \(n - k\) rows (or columns).

424. Show that the rule of signs connecting a cofactor \(A\) with the complementary minor \(M'\) of minor \(M\) can be stated thus: let \(\alpha_1, \alpha_2, \ldots, \alpha_k\) be row numbers, \(\beta_1, \beta_2, \ldots, \beta_l\) column numbers of the minor \(M\) in a determinant \(D\) of order \(n\) written out in increasing order, and let

\[
\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n \quad \text{and} \quad \beta_{k+1}, \beta_{k+2}, \ldots, \beta_l
\]

be, respectively, the numbers of the rows and columns of the complementary minor \(M'\), also written out in the order of increasing magnitude; then \(A = M'\) if the substitution \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) is even, and \(A = -M'\) if the substitution is odd.

Using the Laplace theorem, evaluate the following determinants:

\[
\begin{vmatrix}
5 & 1 & 2 & 7 \\
3 & 0 & 0 & 2 \\
1 & 3 & 4 & 5 \\
2 & 0 & 0 & 3
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & 1 & 3 & 4 \\
2 & 0 & 0 & 2 \\
3 & 0 & 0 & 2 \\
4 & 4 & 7 & 5
\end{vmatrix}
\]

\[
\begin{vmatrix}
0 & 5 & 2 & 0 \\
8 & 3 & 5 & 4 \\
7 & 2 & 4 & 1 \\
4 & 1 & 0 & 4
\end{vmatrix}
\]
### Table 1: Coefficients

<p>| | | | | |</p>
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<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

### Formula

\[ x_1 = \cos \alpha \sin \beta \]
\[ y_1 = \cos \beta \sin \gamma \]
\[ z_1 = \cos \gamma \sin \alpha \]

### Table 2: Transformation Matrix

<p>| | | | |</p>
<table>
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</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

### Equation

\[ a \cdot x + b \cdot y + c \cdot z = d \]
Using the Laplace theorem, compute the following determinants after first transforming them:

\[ \text{450.} \begin{vmatrix} 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \\ 5 & 7 & 9 & 11 \end{vmatrix} \]

\[ \text{451.} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \]

\[ \text{452.} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \]

\[ \text{453.} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \]
454. Isolate, in a determinant $D$ of even order $n = 2k$, four minors $M_1, M_2, M_3, M_4$ of order $k$, as shown below:

$$
\begin{vmatrix}
  a_{11} & \cdots & a_{1k} & a_{1, k+1} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{k1} & \cdots & a_{kk} & a_{k, k+1} & \cdots & a_{kn} \\
  a_{k+1, 1} & \cdots & a_{k+1, k} & a_{k+1, k+1} & \cdots & a_{k+1, n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nk} & a_{n, k+1} & \cdots & a_{nn} \\
\end{vmatrix}
$$

Express $D$ in terms of $M_1, M_2, M_3, M_4$ in the following two cases:

(a) if all elements of $M_2$ or $M_3$ are zero;

(b) if all elements of $M_1$ or $M_4$ are zero.

455. In a determinant $D$ of order $n = kl$, let there be isolated $k$ th-order minors on the secondary diagonal; that is, $M_1$ lies in the first $k$ rows and the last $k$ columns, $M_2$ in the next $k$ rows and the preceding $k$ columns, and so forth, and, finally, $M_l$ lies in the last $k$ rows and the first $k$ columns.

Express $D$ in terms of $M_1, M_2, \ldots, M_l$ if all elements of $D$ on one side of the indicated chain of minors are equal to zero.

456. In a determinant $D$ of order $n$, let there be isolated $k$ rows and $l$ columns, $l \leq k$, and let all elements of the isolated $l$ columns not lying in the isolated $k$ rows be zero. Show that in the Laplace expansion of $D$ in terms of the isolated $k$ rows, one must only take those $k$th-order minors which contain the isolated $l$ columns; this assertion holds true if the rows and columns are interchanged.

457. Use the Laplace theorem to solve problem 206.

458. Prove that

$$
\begin{vmatrix}
  a_{11} & 0 & \cdots & a_{1k} \\
  0 & b_{11} & \cdots & b_{1k} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k1} & 0 & \cdots & b_{kn} \\
  0 & b_{k1} & \cdots & 0 \\
  a_{n1} & 0 & \cdots & 0 \\
  0 & a_{n1} & \cdots & 0 \\
\end{vmatrix}
$$

459. Compute the following determinant of order $k + l$:

$$
\begin{vmatrix}
  3 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
  4 & 3 & 2 & \cdots & 0 & 0 & 0 & 0 \\
  0 & 1 & 3 & 2 & \cdots & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 3 & 2 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\end{vmatrix}
$$

460. Write out the expansion of the following $n$th-order continuant (compare with problem 420):

$$
(a_1, a_2, \ldots, a_n) = \begin{vmatrix}
  a_1 & 1 & 0 & \cdots & 0 & 0 \\
  0 & -1 & a_3 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \cdots & -1 a_n \\
\end{vmatrix}
$$

in terms of the first $k$ rows. What is the property of the Fibonacci numbers (see problem 365) here when $n = 2k$?

461. Prove, without removing parentheses, that the equation

$$(ab' - a'b) (cd' - c'd) - (ac' - a'c) (bd' - b'd)$$

$$+ (ad' - a'd) (bc' - b'c) = 0$$

holds true for arbitrary values of $a, b, c, d, a', b', c', d'$.

462. In the matrix

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  b_{21} & b_{22} & \cdots & b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n1} & b_{n2} & \cdots & b_{nn} \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
  c_{11} & c_{12} & \cdots & c_{1n} \\
  c_{21} & c_{22} & \cdots & c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n1} & c_{n2} & \cdots & c_{nn} \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
  d_{11} & d_{12} & \cdots & d_{1n} \\
  d_{21} & d_{22} & \cdots & d_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  d_{n1} & d_{n2} & \cdots & d_{nn} \\
\end{pmatrix}
$$
containing \( n \) rows and \( 2n \) columns, take any minor \( M \) of order \( n \), containing at least half the columns of the left half of the matrix.

Let \( \sigma \) be the sum of the numbers of the columns of minor \( M \) and let \( M' \) be a minor of order \( n \) made up of the remaining columns of the matrix. Prove that \( \sum (-1)^\sigma M M' = 0 \), where the sum is taken over all minors \( M \) of the indicated type.

*463. Show that the determinants

\[
D = \begin{vmatrix}
|a_{11} x_1, b_{11} x_1, a_{12} x_2, b_{12} x_2, a_{13} x_3, b_{13} x_3, a_{14} x_4, b_{14} x_4|
|a_{21} x_1, b_{21} x_1, a_{22} x_2, b_{22} x_2, a_{23} x_3, b_{23} x_3, a_{24} x_4, b_{24} x_4|
|a_{31} x_1, b_{31} x_1, a_{32} x_2, b_{32} x_2, a_{33} x_3, b_{33} x_3, a_{34} x_4, b_{34} x_4|
|a_{41} x_1, b_{41} x_1, a_{42} x_2, b_{42} x_2, a_{43} x_3, b_{43} x_3, a_{44} x_4, b_{44} x_4|
\end{vmatrix}
\]

\[
\delta = \begin{vmatrix}
\alpha_1 & \beta_1 \\
\alpha_2 & \beta_2 \\
\end{vmatrix}
\text{ and } \Lambda = \begin{vmatrix}
x_1 & x_2 \\
\end{vmatrix}
\begin{vmatrix}
y_1 & y_2 \\
\end{vmatrix}
= \begin{vmatrix}
z_1 & z_2 \\
\end{vmatrix}
\]

are related by the equality \( D = \delta^2 \Delta^2 \) (this property is generalized in problem 530).

*464. Suppose

\[
\begin{align*}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4, \\
g(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4, \\
h(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4,
\end{align*}
\]

and

\[
(x - \alpha) (x - \beta) (x - \gamma) = x^3 + px^2 + qx + r.
\]

Show that

\[
\begin{vmatrix}
\alpha & \alpha^2 \\
\beta & \beta^2 \\
\gamma & \gamma^2 \\
\end{vmatrix}
= \begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 \\
b_0 & b_1 & b_2 & b_3 & b_4 \\
c_0 & c_1 & c_2 & c_3 & c_4 \\
0 & r & p & 1 & 0
\end{vmatrix}
\]

465. We say the determinant

\[
D = \begin{vmatrix}
a_{11} & \ldots & a_{1n} & x_{11} & \ldots & x_{1k} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{n1} & \ldots & a_{nn} & x_{n1} & \ldots & x_{nk} \\
y_{11} & \ldots & y_{kn} & 0 & \ldots & 0 \\
y_{k1} & \ldots & y_{kn} & 0 & \ldots & 0 \\
\end{vmatrix}
\]

is formed by bordering the determinant

\[
\Delta = \begin{vmatrix}
a_{11} & \ldots & a_{1n} \\
a_{21} & \ldots & a_{2n} \\
\vdots & \vdots & \vdots \\
a_{n1} & \ldots & a_{nn} \\
\end{vmatrix}
\]

with \( k \) rows and \( k \) columns.

Show that when \( k \geq n \), \( D = 0 \), and when \( k < n \), \( D \) is a form (that is, a homogeneous polynomial) of degree \( n \) in the elements \( a_{ij} \) of the determinant \( \Delta \) and a form of degree \( 2k \) in the bordering elements \( x_{ij} \), \( y_{ij} \) whose coefficients are the cofactors of \( k \)-th order minors in the determinant \( \Delta \). Namely, prove that \( D = (-1)^{n} \sum A_{1} i_{1} \ldots i_{k} i_{1} \ldots i_{h} \ldots i_{k} X_{i_{1} i_{2}} \ldots i_{h} Y_{i_{1} i_{2}} \ldots i_{k} Y_{i_{1} i_{2}} \ldots i_{k} X_{i_{1} i_{2}} \ldots i_{h} \), where \( A_{i_{1} i_{2}} \ldots i_{h} i_{1} \ldots i_{k} \) is the cofactor of the minor of the determinant \( \Delta \) in the rows labelled \( i_{1}, i_{2}, \ldots, i_{k} \) and in the columns labelled \( j_{1}, j_{2}, \ldots, j_{k} \), while \( X_{i_{1} i_{2}} \ldots i_{h} \) and \( Y_{i_{1} i_{2}} \ldots i_{h} \) are minors of \( D \) made up of bordering elements and lying in the rows (and columns) with the indicated labels. Here the sum is taken over all combinations of indices from \( 1 \) to \( n \) with \( i_{1} < i_{2} < \ldots < i_{k}, j_{1} < j_{2} < \ldots < j_{k} \).

*466. Prove the following generalization of the Laplace theorem. Given an \( m \)-th order determinant \( D \); \( D \) is equal to the sum of all possible products of the type indicated below, if the rows of \( D \) are split into \( \mu \) system without common rows, the first system including rows with labels \( \alpha_{1}, \ldots, \alpha_{\mu} \), the second, rows with labels \( \alpha_{\mu+1}, \ldots, \alpha_{\mu+\nu} \), etc., and, finally, the last \( \nu \) system labels \( \alpha_{2\mu+1} \leq \alpha_{2\mu+2} \leq \ldots \leq \alpha_{3\mu}, \) etc., and if, in the first system of rows we take a minor \( M_{1} \) of order \( k \) in columns with labels \( 1 \leq \beta_{1} \leq \beta_{2} \leq \ldots \leq \beta_{k} \), then in the matrix a minor \( M_{2} \) of order \( k \) lying in columns \( 1 \leq \beta_{1} \leq \beta_{2} \leq \ldots \leq \beta_{k} \) that differs from \( M_{1} \) by the columns of \( M_{1} \), and so on, and, finally, in the \( \nu \)-th
we take a minor $M_p$ of order $s$ lying in the remaining columns with labels $\beta_{n-s+1} < \beta_{n-s+2} < \ldots < \beta_n$; and if we then form the product $eM_1M_2\ldots M_p$, where $e = +1$ if the substitution
\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\beta_1 & \beta_2 & \ldots & \beta_n
\end{pmatrix}
\] (1)

is even, and $e = -1$ if that substitution is odd. That this assertion is a generalization of the Laplace theorem follows from problem 424.

Sec. 7. Multiplication of Determinants

467. Multiply the determinants

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 4
\end{vmatrix}
\begin{vmatrix}
2 & -3 & 1 \\
1 & -4 & 3
\end{vmatrix}
\]

in all four possible ways (by multiplying the rows or columns of the first determinant by the rows or columns of the second one) and verify that in all cases the value of the resulting determinant is equal to the product of the values of the given determinants.

468. Compute the following determinant by squaring it:

\[
\begin{vmatrix}
a & b & c & d \\
-a & a & d & -c \\
-c & d & a & b \\
-d & -c & b & a
\end{vmatrix}
\]

469. Compute the following determinant by squaring it:

\[
\begin{vmatrix}
a & b & c & d & e & f & g & h \\
-b & a & d & -c & f & -e & -h & g \\
-c & d & a & b & h & -e & -f & c \\
-d & c & b & a & h & -g & f & e \\
-e & -f & -g & -h & a & b & c & d \\
-f & e & -h & g & b & a & d & c \\
-g & h & e & -f & c & d & a & b \\
-h & -g & f & e & d & c & b & a
\end{vmatrix}
\]

Evaluate the following determinants by representing them as products of determinants:

\[
\begin{pmatrix}
1 + x_1y_1 & 1 + x_1y_2 & \ldots & 1 + x_1y_n \\
1 + x_2y_1 & 1 + x_2y_2 & \ldots & 1 + x_2y_n \\
\vdots & \vdots & \ddots & \vdots \\
1 + x_ny_1 & 1 + x_ny_2 & \ldots & 1 + x_ny_n
\end{pmatrix}
\]

\[
\begin{vmatrix}
\cos(a_1 - \beta_1)
\cos(a_1 - \beta_1)
\cos(a_1 - \beta_1)
\cos(a_n - \beta_1)
\cos(a_n - \beta_1)
\cos(a_n - \beta_1)
\end{vmatrix}
\]

\[
\begin{vmatrix}
\sin(a_1 + \alpha_1)
\sin(a_1 + \alpha_1)
\sin(a_2 + \alpha_1)
\sin(a_n + \alpha_1)
\sin(a_2 + \alpha_1)
\sin(a_2 + \alpha_1)
\end{vmatrix}
\]

\[
\begin{vmatrix}
(\alpha_1 + \beta_1)^n & (\alpha_1 + \beta_1)^n & \ldots & (\alpha_1 + \beta_1)^n \\
(\alpha_2 + \beta_1)^n & (\alpha_2 + \beta_1)^n & \ldots & (\alpha_2 + \beta_1)^n \\
(\alpha_n + \beta_1)^n & (\alpha_n + \beta_1)^n & \ldots & (\alpha_n + \beta_1)^n
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 + 2^{n-1} & 2^{n-1} & \ldots & 1 \\
2^{n-1} & 3^{n-1} & \ldots & (n + 1)^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1^{n-1} & (n + 1)^{n-1} & \ldots & (2n - 1)^{n-1}
\end{vmatrix}
\]
477. \[
\begin{array}{cccc}
s_0 & s_1 & s_2 & \ldots & s_{n-1} \\
s_1 & s_2 & s_3 & \ldots & s_n \\
s_2 & s_3 & s_4 & \ldots & s_{n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{n-1} & s_n & s_{n+1} & \ldots & s_{2n-2} \\
\end{array}
\]
where \( s_n = x_1^k + x_2^k + \ldots + x_n^k \).

*478. \[
\begin{array}{cccc}
s_0 & s_1 & s_2 & \ldots & s_{n-1} \\
s_1 & s_2 & s_3 & \ldots & s_n \\
s_2 & s_3 & s_4 & \ldots & s_{n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{n-1} & s_n & s_{n+1} & \ldots & s_{2n-2} \\
\end{array}
\]
where \( s_n = x_1^2 + x_2^2 + \ldots + x_n^2 \).

*479. Prove that the value of the circulant is given by the equality
\[
\begin{vmatrix}
a_1 & a_2 & a_3 & \ldots & a_n \\
a_2 & a_3 & a_4 & \ldots & a_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1} & a_n & a_1 & \ldots & a_{n-2} \\
a_n & a_1 & a_2 & \ldots & a_{n-1} \\
\end{vmatrix} = \frac{1}{n} \prod_{1 \leq i < j \leq n} (\xi_i - \xi_j),
\]
where \( f(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1} \) and \( \xi_1, \xi_2, \ldots, \xi_n \) are all values of the \( n \)th root of unity.

480. Prove, using the notation of problem 479, that
\[
\begin{vmatrix}
a_1 & a_2 & a_3 & \ldots & a_n \\
a_2 & a_3 & a_4 & \ldots & a_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1} & a_n & a_1 & \ldots & a_{n-2} \\
a_n & a_1 & a_2 & \ldots & a_{n-1} \\
\end{vmatrix} = \left( -1 \right)^{\frac{(n-1)(n-2)}{2}} \prod_{1 \leq i < j \leq n} (\xi_i - \xi_j).
\]

*481. Compute the determinant
\[
\begin{vmatrix}
1 & \alpha & \alpha^2 & \ldots & \alpha^{n-1} \\
\alpha & 1 & \alpha & \ldots & \alpha^{n-2} \\
\alpha & \alpha & 1 & \ldots & \alpha^{n-3} \\
\alpha & \alpha & \alpha & \ldots & 1 \\
\end{vmatrix}
\]

482. Compute the following determinant using the result of problem 479:
\[
\begin{vmatrix}
a & b & \cdots & b \\
b & a & \cdots & b \\
\vdots & \vdots & \ddots & \vdots \\
b & b & \cdots & a \\
\end{vmatrix}
\]

483. Use the result of problem 479 to compute the determinant
\[
\begin{vmatrix}
a & b & c & d \\
d & a & b & c \\
c & d & a & b \\
\end{vmatrix}
\]

Evaluate the following determinants:

484. \[
\begin{vmatrix}
1 & C_n^1 & C_n^2 & \ldots & C_n^{n-1} \\
1 & 1 & C_n^1 & \ldots & C_n^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_n^{n-1} & 1 & 1 & \ldots & C_n^{n-2} \\
C_n^1 & C_n^2 & C_n^3 & \ldots & 1 \\
\end{vmatrix}
\]

485. \[
\begin{vmatrix}
1 & 2a & 3a^2 & \ldots & ma^{n-1} \\
ma^{n-1} & 1 & 2a & \ldots & (n-1)a^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2a & 3a^2 & 4a^3 & \ldots & 1 \\
\end{vmatrix}
\]

*486. Prove that
\[
\begin{vmatrix}
s-a_1 & s-a_2 & \ldots & s-a_n \\
s-a_2 & s-a_3 & \ldots & s-a_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
s-a_n & s-a_1 & \ldots & s-a_{n-2} \\
\end{vmatrix} = \frac{1}{n-1} (n-1) a_1 a_2 \ldots a_n
\]
where \( s = a_1 + a_2 + \ldots + a_n \).
Compute the following determinants:

\[
\begin{vmatrix}
1 -1 & -1 & \cdots & -1 & 1 \\
-1 & 1 -1 & \cdots & -1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & 1 -1 & 1
\end{vmatrix}
\]

\[
\begin{vmatrix}
a a a \cdots a \\
b b b \cdots b \\
b b b \cdots b \\
a a a \cdots b \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\cos \frac{\pi}{n} & \cos \frac{2\pi}{n} & \cos \frac{3\pi}{n} & \cdots & -1 \\
-1 & \cos \frac{\pi}{n} & \cos \frac{2\pi}{n} & \cdots & \cos \frac{(n-2)\pi}{n} \\
\cos \frac{(n-1)\pi}{n} & -1 & \cos \frac{\pi}{n} & \cdots & \cos \frac{(n-2)\pi}{n} \\
\cos \frac{2\pi}{n} & \cos \frac{3\pi}{n} & \cos \frac{4\pi}{n} & \cdots & \cos \frac{\pi}{n}
\end{vmatrix}
\]

\[
\begin{vmatrix}
sin \frac{\pi}{n} & sin \frac{(2+2h)}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+6h)}{n} & \sin \frac{(2+8h)}{n} & \sin \frac{(2+10h)}{n} & \sin \frac{(2+12h)}{n} \\
\sin \frac{(2+2h)}{n} & sin \frac{\pi}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+6h)}{n} & \sin \frac{(2+8h)}{n} & \sin \frac{(2+10h)}{n} & \sin \frac{(2+12h)}{n} \\
\sin \frac{(2+4h)}{n} & \sin \frac{(2+2h)}{n} & sin \frac{\pi}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+6h)}{n} & \sin \frac{(2+8h)}{n} & \sin \frac{(2+10h)}{n} \\
\sin \frac{(2+6h)}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+2h)}{n} & sin \frac{\pi}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+6h)}{n} & \sin \frac{(2+10h)}{n} \\
\sin \frac{(2+8h)}{n} & \sin \frac{(2+6h)}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+2h)}{n} & sin \frac{\pi}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+10h)}{n} \\
\sin \frac{(2+10h)}{n} & \sin \frac{(2+8h)}{n} & \sin \frac{(2+6h)}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+2h)}{n} & sin \frac{\pi}{n} & \sin \frac{(2+12h)}{n} \\
\sin \frac{(2+12h)}{n} & \sin \frac{(2+10h)}{n} & \sin \frac{(2+8h)}{n} & \sin \frac{(2+6h)}{n} & \sin \frac{(2+4h)}{n} & \sin \frac{(2+2h)}{n} & sin \frac{\pi}{n}
\end{vmatrix}
\]

\[
\begin{vmatrix}
1^2 & 1^2 & 1^2 & \cdots & (n-1)^2 \\
2^2 & 2^2 & 2^2 & \cdots & (n-1)^2 \\
3^2 & 3^2 & 3^2 & \cdots & (n-1)^2 \\
(1^2)^2 & (2^2)^2 & (3^2)^2 & \cdots & (n-1)^2 \\
2^2 & 3^2 & 4^2 & \cdots & 1^2
\end{vmatrix}
\]

\[
\begin{vmatrix}
a_1 & a_2 & a_3 & \cdots & a_n \\
-a_n & a_1 & a_2 & \cdots & a_{n-1} \\
-a_{n-1} & -a_n & a_1 & \cdots & a_{n-2} \\
-\cdots & -\cdots & -\cdots & \cdots & -\cdots \\
-a_2 & -a_3 & -a_4 & \cdots & a_1
\end{vmatrix}
\]

where \( z \) is any number.

*495. Prove that a circulant of order \( 2n \) with the first row made up of the elements \( a_1, a_2, \ldots, a_{2n} \) is equal to the product of a circulant of order \( n \) with the first row made up of elements \( a_1 + a_{n+1}, a_2 + a_{n+2}, \ldots, a_n + a_{2n} \) and a skew circulant of order \( n \) with the first row made up of the elements \( a_1 - a_{n+1}, a_2 - a_{n+2}, \ldots, a_n - a_{2n} \).

*496. Multiply the two determinants

\[
\begin{vmatrix}
x_1 & x_2 & x_3 & x_4 \\
x_2 & x_3 & x_4 & x_5 \\
x_3 & x_4 & x_5 & x_6 \\
x_4 & x_5 & x_6 & x_7
\end{vmatrix}
\begin{vmatrix}
y_1 & y_2 & y_3 & y_4 \\
y_2 & y_3 & y_4 & y_5 \\
y_3 & y_4 & y_5 & y_6 \\
y_4 & y_5 & y_6 & y_7
\end{vmatrix}
\]

to prove the Euler identity:

\[
(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2 + (x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2)^2 + (x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1)^2.
\]

What property of the integers follows from this?

*497. Use multiplication of determinants to prove the identity

\[
(a^3 + b^3 + c^3 - 3abc)(a^3 + b^3 + c^3 - 3abc) = a^3 + b^3 + c^3 - 3abc.
\]
*498. Using the notation of problem 497, prove the following identity:

\[ (a^2 + b^2 + c^2 - ab - ac - bc) \times (a'^2 + b'^2 + c'^2 - a'b' - a'c' - b'c') = A^2 + B^2 + C^2 - AB - AC - BC. \]

*499. Prove the following generalization of the theorem on multiplication of determinants: given two matrices

\[
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix} \quad \text{and} \quad
\begin{pmatrix}
    b_{11} & b_{12} & \ldots & b_{1n} \\
    b_{21} & b_{22} & \ldots & b_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{m1} & b_{m2} & \ldots & b_{mn}
\end{pmatrix}
\]

each having \( m \) rows and \( n \) columns.

Combining the rows of one matrix with the rows of the other, putting \( c_{ij} = \sum_{h=1}^{n} a_{ih}b_{jh} \), set up an \( m \)-order determinant:

\[ D = \begin{vmatrix}
    c_{11} & c_{12} & \ldots & c_{1m} \\
    c_{21} & c_{22} & \ldots & c_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{m1} & c_{m2} & \ldots & c_{mm}
\end{vmatrix} \]

Denote by \( A_{i_1i_2\ldots i_m} \) and \( B_{i_1i_2\ldots i_m} \) respectively the \( m \)-order minors of matrices \( A \) and \( B \) made up of the columns (of these matrices) labelled \( i_1, i_2, \ldots, i_m \) in the same order. Then

\[ D = \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} A_{i_1i_2\ldots i_m}B_{i_1i_2\ldots i_m} \quad (1) \]

for \( m \leq n \) (the Binet-Cauchy formula), that is, the determinant \( D \) is equal to the sum of the products of all \( m \)-order minors of matrix \( A \) into the corresponding minors of matrix \( B \). When \( m > n \),

\[ D = 0. \quad (2) \]

*501. Prove the following Cauchy identity without carrying out the multiplication:

\[ (a_1c_1 + a_2c_2 + \ldots + a_nc_n)(b_1d_1 + b_2d_2 + \ldots + b_nd_n) - (a_1d_1 + a_2d_2 + \ldots + a_nd_n)(b_1c_1 + b_2c_2 + \ldots + b(nc_n) = \sum_{1 \leq i < j \leq n} (a_ib_j - a_jb_i)(c_id_j - c_jd_i) \quad (n > 1). \]

*502. Prove the Lagrange identity without multiplying:

\[ \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) - \left( \sum_{i=1}^{n} a_i b_i \right)^2 = \sum_{1 \leq i < j \leq n} (a_ib_j - a_jb_i)^2. \]

*503. Prove that for any real numbers \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) the following inequality holds true:

\[ (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \geq (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2. \]

Note that the equality sign holds if and only if one of the given sets of numbers differs from the other solely by a numerical factor (which may be zero). (The Cauchy-Bunyakovsky inequality.)

*504. Prove that for any complex numbers \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) the following equality holds:

\[ \left( \sum_{h=1}^{n} a_h \overline{a_h} \right) \left( \sum_{h=1}^{n} b_h \overline{b_h} \right) - \left( \sum_{h=1}^{n} a_h \overline{b_h} \right) \left( \sum_{h=1}^{n} \overline{a_h} b_h \right) = \sum_{1 \leq i < j \leq n} (a_ib_j - a_jb_i)(a_jb_i - a_ib_j). \]

*505. Prove that for any two sets of complex numbers \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) the following inequality holds true:

\[ \left( \sum_{h=1}^{n} |a_h|^2 \right) \left( \sum_{h=1}^{n} |b_h|^2 \right) \geq \left( \sum_{h=1}^{n} a_h \overline{b_h} \right)^2. \]

Note that the equality sign holds if and only if the numbers of one of the given sets differ from those of the other by a numerical factor alone.

*506. The adjugate determinant of a determinant \( D \) of order \( n > 1 \) is a determinant \( D' \) obtained from \( D \) by re-
Show that

\[ D' = D^{-1}. \] (1)

*507. Let \( M \) be an \( m \)th-order minor of a determinant \( D \), let \( A \) be the cofactor of \( M \), let \( M' \) be a minor of the adjugate determinant \( D' \) corresponding to the minor \( M \) (that is, made up of the cofactors of the elements of \( D \) that enter into \( M \)).

Prove the equality \( M' = D^{m-1} A \).

If we agree to regard the complementary minor of the entire determinant \( D \) as equal to 1, then this equality will be a generalization of the equality of problem 506 (for \( m = n \)).

*508. Let \( C \) be a minor of the \((n - 2)\)th order obtained from the determinant \( D \) by crossing out the \( i \)th and \( j \)th rows and the \( k \)th and \( l \)th columns with \( i < j \) and \( k < l \); \( A_{pq} \) is, as usual, the cofactor of the element \( a_{pq} \).

Prove that

\[
\begin{vmatrix}
A_{ik} & A_{ij} \\
A_{jk} & A_{lj}
\end{vmatrix} = (-1)^{i+j+k+l} DC.
\]

*509. Show that if the determinant \( D \) is equal to zero then all the rows (and also the columns) of the adjugate determinant are proportional.

*510. Let \( a_{ij} \) be an element of determinant \( D \) of order \( n \) and let \( A_{pq} \) be the cofactor of the corresponding element \( a_{ij} \) of determinant \( D' \), the adjugate of \( D \). Show that \( A_{ij} A_{pq} = D^{n-1} a_{ij} \).

*511. Let \( M \) be a minor of order \( m \) of the determinant \( D \) of order \( n \); let \( A' \) be the minor, corresponding to \( M \), of the adjugate determinant \( D' \) and let \( A' \) be the cofactor of the minor \( M' \). Prove that \( A' = D^{n-m-1} A \). This is a generalization of the equality of problem 500.

*512. Knowing the minors of all elements of a nonzero determinant \( D \), find the elements.

**513.** Let

\[
s_1 s_2 \ldots s_n \quad \text{and} \quad s_{n+1} s_{n+2} \ldots s_{2n},
\]

\[ p = s_1 s_2 \ldots s_n. \]

Show that

\[
\begin{vmatrix}
s_1 & s_2 & \ldots & s_n \\
s_2 & s_3 & \ldots & s_{n+1} \\
\vdots & \vdots & & \vdots \\
s_n & s_{n+1} & \ldots & s_{2n}
\end{vmatrix}
= p^2
\]

514. Show that if

\[
D(x) = \begin{vmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} - x & \ldots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn} - x
\end{vmatrix}
\]

then the product \( D(x) \cdot D(-x) \) can be expressed as

\[
\begin{vmatrix}
A_{11} - x^2 & A_{12} & \ldots & A_{1n} \\
A_{21} & A_{22} - x^2 & \ldots & A_{2n} \\
\vdots & \vdots & & \vdots \\
A_{n1} & A_{n2} & \ldots & A_{nn} - x^2
\end{vmatrix}
\]

where all the \( A_{ij} \) are independent of \( x \). Find the expression of \( A_{ij} \) in terms of \( a_{ij} \).

*515. Use multiplication of determinants to prove that a determinant reverses sign under an interchange of two rows (or two columns).

*516. Use multiplication of determinants to prove that a determinant remains unchanged if a row or column multiplied by a number \( c \) is added to another row or column.

*517. Show that the determinant

\[
\begin{vmatrix}
1 & \cos q_1 & \cos q_2 \\
\cos q_1 & 1 & \cos q_1 \\
\cos q_2 & \cos q_1 & 1
\end{vmatrix}
\]

is zero if \( q_1 \cdot q_2 \cdot q_3 = 0 \).

*518. Let \( l_1, l_2, l_3 \) and \( m_1, m_2, m_3 \), \( m_3 \) be the angles of two rays with the orthogonal coordinate axes and let \( q \) be the angle between the rays. Prove that

\[
s_1 l_1 m_1^2 + s_2 l_2 m_2^2 + s_3 l_3 m_3^2 - s_1 l_1 m_1 l_2 m_2 + s_2 l_2 m_2 l_3 m_3 - s_3 l_3 m_3 l_1 m_1 = 0.
\]

519. Let \( x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3 \) be the angles of three rays \( l_1, l_2, l_3 \) with the orthogonal coordinate axes and let the angles between the rays be \( q_1 = \mu_1 = \mu_2 \).

6*
\[
\theta_2 = \angle (L_2, L_1), \quad \theta_3 = \angle (L_1, L_2). \text{ Prove that:} \n\]
\[
\cos \alpha_1 \cos \beta_1 \cos \gamma_1 \quad \cos \alpha_2 \cos \beta_2 \cos \gamma_2 \quad \cos \alpha_3 \cos \beta_3 \cos \gamma_3
\]
\[
= 1 - \cos^2 \theta_1 - \cos^2 \theta_2 - \cos^2 \theta_3 + 2 \cos \theta_1 \cos \theta_2 + 2 \cos \theta_1 \cos \theta_3 + 2 \cos \theta_2 \cos \theta_3.
\]

*520. Let \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) be the rectangular coordinates of the points \(M_1, M_2, M_3\) in the plane. Show that the determinant
\[
\begin{vmatrix}
 x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1
\end{vmatrix}
\]
remains unchanged under a rotation of the coordinate axes and a translation of the origin. Use this fact to elucidate the geometric meaning of the determinant.

*521. Let \((x_1, y_1)\) and \((x_2, y_2)\) be the rectangular coordinates of two points \(M_1, M_2\) in the plane. After determining the geometric meaning of the determinant \(\begin{vmatrix}
 x_1 & y_1 \\
x_2 & y_2
\end{vmatrix}\), figure out whether it undergoes change under a rotation of the axes and a translation of the coordinate origin.

*522. Compute the product of the determinants
\[
\begin{vmatrix}
 x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
x_3 & y_3 & z_3
\end{vmatrix}
\]
and thus obtain an expression of the radius of a circumscribed circle in terms of the sides \(a, b, c\) and the area \(S\) of a triangle.

*523. Let \(\beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3\) be, respectively, the cosines of the angles of three pairwise orthogonal rays \(OA, OB, OC\) with axes \(Ox, Oy, Oz\) of a rectangular coordinate system. Prove that the determinant
\[
\begin{vmatrix}
 \beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{vmatrix}
\]
remains unchanged under a rotation of the coordinate axes and a translation of the origin. Use this fact to elucidate the geometric meaning of the determinant.

*524. Let \((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\) be the rectangular coordinates of three points \(M_1, M_2, M_3\) in space. Determine the geometric meaning of the determinant
\[
\begin{vmatrix}
 x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
x_3 & y_3 & z_3
\end{vmatrix}
\]
and thus obtain an expression of the radius of a circumscribed sphere in terms of the sides \(a, b, c\) and the volume \(V\) of a tetrahedron.

*525. Find an expression for the volume \(V\) of a parallelepiped in terms of the lengths \(a, b, c\) of its edges passing through a single vertex and the angles \(\alpha, \beta, \gamma\) formed by the edges. (The angle \(\alpha\) is formed by \(b\) and \(c\); \(\beta\) by \(c\) and \(a\); and \(\gamma\) by \(a\) and \(b\).)

*526. Let \(l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3\) be the cosines of the angles of rays \(OA, OB, OC\) with the positive semi-axes \(Ox, Oy, Oz\) of a rectangular coordinate system. Prove that for the rays \(OA, OB\) and \(OC\) to be coplanar, it is necessary and sufficient that
\[
\begin{vmatrix}
 l_1 & l_2 & l_3 \\
m_1 & m_2 & m_3 \\
n_1 & n_2 & n_3
\end{vmatrix} = 0.
\]

*527. Let \((x_i, y_i, z_i)\) be the rectangular coordinates of a point \(M_i\) of space \((i = 1, 2, 3, 4)\). Determine the geometric meaning of the determinant
\[
\begin{vmatrix}
 x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
x_3 & y_3 & z_3 \\
x_4 & y_4 & z_4
\end{vmatrix}
\]
by showing that it does not change under a translation of the origin.
528. Multiply together the determinants
\[
\begin{vmatrix}
x_1 & y_1 & z_1 & \cdots & x_r \cdot y_r & z_r \\
x_2 & y_2 & z_2 & \cdots & x_r & y_r & z_r \\
x_3 & y_3 & z_3 & \cdots & x_r & y_r & z_r \\
x_4 & y_4 & z_4 & \cdots & x_r & y_r & z_r \\
\end{vmatrix}
\]
and
\[
\begin{vmatrix}
x_1 & -y_1 & -z_1 & \cdots & -x_r & -y_r & -z_r \\
x_2 & -y_2 & -z_2 & \cdots & -x_r & -y_r & -z_r \\
x_3 & -y_3 & -z_3 & \cdots & -x_r & -y_r & -z_r \\
x_4 & -y_4 & -z_4 & \cdots & -x_r & -y_r & -z_r \\
\end{vmatrix}
\]
to obtain an expression for the radius of a sphere circumscribed about an arbitrary tetrahedron in terms of the volume and an edge of the tetrahedron. In particular, use the expression obtained to find the radius of a sphere circumscribed about a regular tetrahedron with the length of an edge \(a\).

Sec. 8. Miscellaneous Problems

529. Show that a determinant of order \(n\) permits of the following axiomatic definition (which is equivalent to the ordinary definition).

Any row of \(n\) numbers (or of the elements of any field \(P\)) will be called a vector and will be denoted by a single letter in boldface type. The addition of two vectors and the multiplication of a vector by a scalar (number) are defined as usual, that is, if
\[
a = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad b = (b_1, b_2, \ldots, b_n)
\]
then
\[
a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n),
\]
if \(c\) is a scalar, then
\[
c a = (ca_1, ca_2, \ldots, ca_n).
\]
A function \(f(a_1, a_2, \ldots, a_n)\) of \(n\) vectors with numerical values is termed a linear function in each argument (or, simply, polylinear) if
\[
f(a_1, \ldots, c' a_i + c'' a_i, \ldots, a_n) = c' f(a_1, \ldots, a_i, \ldots, a_n) + c'' f(a_1, \ldots, a_i, \ldots, a_n) \quad (\alpha)
\]
for any vectors, any numbers \(c', c''\) and any \(i = 1, 2, \ldots, n\). Also we say that a function has the property of annihilation if
\[
f(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = 0 \quad \text{for} \quad a_i = a_j \quad (\beta)
\]
for \(i, j = 1, 2, \ldots, n, \quad i \neq j\).

Let \(e_i\) (\(i = 1, 2, \ldots, n\)) be a vector with unity in the \(i\)th position and zeros elsewhere. The function \(f(a_1, a_2, \ldots, a_n)\) is said to be normalized if
\[
f(e_1, e_2, \ldots, e_n) = 1
\]
Given a square matrix of order \(n\),
\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\]
and, in the ordinary sense, its determinant \(|A|\), that is,
\[
|A| = \sum (-1)^s a_{1i_1} a_{2i_2} \cdots a_{ni_n}
\]
where the sum is taken over all permutations \(i_1, i_2, \ldots, i_n\) of the numbers \(1, 2, \ldots, n\), and \(s\) is the number of inversions in each permutation.

Show that
(1) the determinant \(|A|\), as a function of the rows of matrix \(A\), has the properties (\(\alpha\)), (\(\beta\)), (\(\gamma\));
(2) any function of \(n\) vectors having the properties (\(\alpha\)) and (\(\beta\)) satisfies the equality
\[
f(a_1, a_2, \ldots, a_n) = |A| f(e_1, e_2, \ldots, e_n),
\]
where \(A\) is a matrix with rows \(a_1, a_2, \ldots, a_n\);
(3) any function \(f(a_1, a_2, \ldots, a_n)\) having the properties (\(\alpha\)), (\(\beta\)) and (\(\gamma\)) is equal to the determinant \(|A|\) of matrix \(A\) with the rows \(a_1, a_2, \ldots, a_n\). In other words, the determinant \(|A|\) of matrix \(A\) is the sole polylinear, normalized function of its rows with the property of annihilation.

530. Use assertion (2) of the preceding problem to prove the theorem of the multiplication of determinants.

531. Show that for functions of \(n\) vectors over a field with characteristic different from 2, the property (\(\beta\)), given property (\(\alpha\)), is equivalent to the alternating-sign property of the function, that is,
\[
f(a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots, a_n) =
\]
\[
= -f(a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots, a_n) \quad (\beta')
\]
for arbitrary vectors and arbitrary \(i, j = 1, 2, \ldots, n, \quad i \neq j\). Construct an example of a function of \(n\) vectors over a field \(P\) with characteristic 2 having the properties (\(\alpha\)), (\(\beta'\)) and (\(\gamma\)), but not having the property (\(\beta\)).
532. Compute the determinant
\[
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & e & e^2 & \cdots & e^{n-1} \\
1 & e & e^2 & \cdots & e^{2(n-1)} \\
1 & e & e^2 & \cdots & e^{3(n-1)} \\
1 & \cdots & \cdots & \cdots & \cdots \\
1 & e \cdot e^{n-1} & e^{2(n-1)} & \cdots & e^{(n-1)n}
\end{vmatrix}
\]
where \( e = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \).

533. What changes does a determinant undergo if we isolate \( k \) rows (or columns) and subtract from each one the remaining isolated rows?

534. Express the determinant
\[
D = \begin{vmatrix}
a_{11} + x & a_{12} + x & \cdots & a_{1n} + x \\
a_{21} + x & a_{22} + x & \cdots & a_{2n} + x \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} + x & a_{n2} + x & \cdots & a_{nn} + x
\end{vmatrix}
\]
as a polynomial arranged in powers of \( x \).

535. Prove that the sum of the cofactors of all elements of the determinant
\[
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]
is equal to the determinant
\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & a_{21} - a_{11} & a_{22} - a_{12} & \cdots & a_{2n} - a_{1n} \\
1 & a_{31} - a_{11} & a_{32} - a_{12} & \cdots & a_{3n} - a_{1n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & a_{n1} - a_{11} & a_{n2} - a_{12} & \cdots & a_{nn} - a_{1n}
\end{vmatrix}
\]

536. Prove that the sum of the cofactors of all elements of a determinant remains unchanged if the same number is added to all elements.

537. Prove that if all elements of some row (or column) of a determinant are equal to unity, then the sum of the cofactors of all elements of the determinant is equal to the determinant itself.

538. Prove that a skew-symmetric determinant of even order remains unchanged if the same number is added to all elements.

539. Establish the following relationship of continuant (see problem 420 for an expanded expression of a continuant)
\[
\begin{vmatrix}
a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & a_3 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & a_n & 1
\end{vmatrix}
\]
and continued fractions
\[
a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}}.
\]

540. Given two determinants:
\[
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]
of order \( n \)
and
\[
\begin{vmatrix}
h_{11} & h_{12} & \cdots & h_{1r} \\
h_{21} & h_{22} & \cdots & h_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
h_{r1} & h_{r2} & \cdots & h_{rr}
\end{vmatrix}
\]
of order \( r \).
Form a determinant of order \( np \):

\[
D = 
\begin{bmatrix}
  a_{11}b_{11} & a_{12}b_{12} & \ldots & a_{1n}b_{1n} \\
  a_{21}b_{21} & a_{22}b_{22} & \ldots & a_{2n}b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1}b_{n1} & a_{n2}b_{n2} & \ldots & a_{nn}b_{nn}
\end{bmatrix}
\]

Thus the matrix of the determinant \( D \) consists of \( p^2 \) blocks with \( n \) rows and \( n \) columns in each. Here, the block in the \( i \)th row block and the \( j \)th column block (for arbitrary \( i, j = 1, 2, \ldots, p \)) is obtained from the matrix of determinant \( A \) by multiplying all its elements by \( b_{ij} \). Prove that \( D = A^pB^n \). The determinant \( D \) is said to be the Kronecker product of the determinants \( A \) and \( B \) (see problems 963, 965).

541. Prove the following rule for expanding a bordered determinant: if

\[
D = 
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix}
\]

and \( A_{ij} \) is the cofactor of element \( a_{ij} \), then

\[
D = \sum_{i,j=1}^{n} A_{ij}x_iy_j
\]

542. Suppose the elements of a determinant \( D \) are polynomials of the unknowns \( x_1, x_2, \ldots, x_n \) with numerical coefficients (coefficients of an arbitrary field \( P \)), and \( b_{ij} = 0 \). Prove that it is possible to represent the cofactors

of the elements of \( D \) in the form \( A_{ij} = A_{i}B_{j}, i, j = 1, 2, \ldots, n \), where all the \( A_i \) and \( B_j \) are polynomials of \( x_1, x_2, \ldots, x_n \). Find these polynomials for the determinant \( \Delta \):

\[
\Delta = 
\begin{bmatrix}
  0 & a & b \\
  -a & 0 & c \\
  -b & -c & 0
\end{bmatrix}
\]

where the unknowns are \( a, b, c \).

*543. Use the two preceding problems to prove that a skew-symmetric determinant of even order is the square of some polynomial of its elements that lie above the principal diagonal.

*544. Show that if in the general expression of a skew-symmetric determinant every element \( a_{ij} \) for \( j > i \) is replaced by \( -a_{ij} \), then all terms will cancel whose substitutions of the \( n \) decomposed into cycles) yield at least one cycle of odd length.

*545. Let \( D \) be a skew-symmetric determinant of even order \( n \) with elements \( a_{ij} = -a_{ji} (i, j = 1, 2, \ldots, n) \). The Pfaffian product of determinant \( D \) is the product

\[
\varepsilon_{a_1, a_2, \ldots, a_n} a_{a_1a_2} a_{a_3a_4} \ldots a_{a_{n-1}a_n}
\]

in which the indices of \( n/2 \) component elements form a permutation \( a_1, a_2, \ldots, a_n \) of the numbers 1, 2, \ldots, \( n \); \( \varepsilon = +1 \) if the permutation is even and \( \varepsilon = -1 \) if the permutation is odd. The Pfaffian product is said to be reduced if it consists solely of elements above the principal diagonal in \( D \) (that is, if the first index of each element is less than the second index). We will use the term essential to describe a term of determinant \( D \) if the substitution of its indices has cycles of even length only. A pair of reduced Pfaffian products \( N_1, N_2 \) (in a given order) is said to correspond to a given essential term of \( D \) if it is constructed on the basis of that term in the following manner. Suppose a substitution of indices of a given term is written cyclically thus:

\[
(c_1a_{c_1}, c_2a_{c_2}, \ldots, c_n) (b_1, b_2, \ldots, b_n) (k_1, k_2, \ldots, k_n)
\]

Here, \( c_i = 1 \) and each cycle, from the second onwards, begins with the smallest number of numbers that do not appear in the preceding cycles. We now construct the Pfaffian products \( N_1 = c_1a_{c_1}a_{c_2}a_{c_3} \ldots a_{c_{n-1}} a_{c_n} b_{b_1} a_{b_2} b_{b_3} \ldots b_{b_{n-1}} b_{b_n} \)

\[
N_2 = b_{b_1} a_{b_2} a_{b_3} \ldots a_{b_{n-1}} a_{b_n} c_1 a_{c_2} a_{c_3} \ldots c_{n-1} c_n
\]
and then each element $a_{ij}$, where $i > j$, is replaced by $-a_{ij}$.

Accordingly, the sign of $e_1$ or $e_2$ is changed, but the substitution class is also changed so that for each replacement we again obtain a Pfaffian product. Thus, by carrying out in $N_1'$ and $N_1''$ all the indicated replacements, we obtain the pair $N_1$, $N_2$ of reduced Pfaffian products that corresponds to the given essential term of $D$.

Prove that

(1) any pair of reduced Pfaffian products (distinct or identical) corresponds to one and only one essential term of the general expansion of determinant $D$. (In the general expansion of $D$, terms obtained from one another by replacements of the type $a_{ij} = -a_{ji}$ are regarded as distinct.) In other words, a one-to-one correspondence has been established between all essential terms and all pairs of reduced Pfaffian products of the determinant $D$;

(2) each essential term is equal to the product of reduced Pfaffian products of the corresponding pair;

(3) $D = p^2$, where $p$ is the sum of all reduced Pfaffian products; it is called the Pfaffian aggregate, or the Pfaffian of the determinant $D$.

*549. Let $D$ be a skew-symmetric determinant of order $n$ with elements $a_{ij}$ ($i, j = 1, 2, \ldots, n$). Minor of the element $a_{ij}$, $p_{i, n+1}$ is the Pfaffian of the minor $M_{ij}$. Show that $M_{n+1} = p_{i+1, n+1} = (-1)^{i+1} p_{i, n+1}$ (provided that the unknowns $e_1, \ldots, e_n$ are elements of $D$ lying above the principal diagonal).

Verify that in this manner we obtain for the determinant

\[
\begin{vmatrix}
0 & a & b & c \\
-a & 0 & c & b \\
b & -c & 0 & a \\
c & b & -a & 0
\end{vmatrix}
\]

the same result as in problem 542 (if one takes into account that in problem 542 $A_i$ and $B_j$ are determined to within change of sign in all the polynomials).

*550. Prove that a determinant of general form, regarded as a polynomial of its elements assumed to be the unknowns, cannot be decomposed into two factors, each of which is a polynomial in the same unknowns of a nonzero degree. In other words, the determinant is an irreducible polynomial of its elements and, what is more, over any field.

*551. Let $D = | a_{ij} |$ be a determinant of order $n > 1$ and let $k$ be any one of the numbers $1, 2, \ldots, n$.

\[
C_k = \frac{n!}{k!(n-k)!}
\]

Denote by $s_1, s_2, \ldots, s_{n+1}$ all possible combinations of $n$ numbers $1, 2, \ldots, n$, taken $k$ at a time, that are numbered in an arbitrary (but, from then on, invariable) order. For the sake of definiteness, the numbers in each combination can be assumed to be arranged in the order of increasing magnitude, although this is not essential for what follows. Let $\mu_{ij}$ be the $k$th-order minor of determinant $D$ at the intersection of rows with labels taken from the combination $s_i$ and columns with labels taken from the combination $s_j$, where $i, j = 1, 2, \ldots, n$.

Let $a_{ij}$ be the cofactor of the minor $\mu_{ij}$ in $D$. We use the term determinant of minors of order $k$ of the determi
nant $D$ for a determinant of order $\binom{n}{k}$ having the form

$$
\Delta_k = \begin{vmatrix}
\xi_{11} & \xi_{12} & \cdots & \xi_{1k} \\
\xi_{21} & \xi_{22} & \cdots & \xi_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{n1} & \xi_{n2} & \cdots & \xi_{nk}
\end{vmatrix}
$$

We introduce yet another determinant, $\Delta_k$ of order $\binom{n}{k}$, that is obtained from $\Delta_k$ by replacing each minor $\xi_{ij}$ by its cofactor $\alpha_{ij}$ in $D$.

Prove that

(1) The values of the determinants $\Delta_k$ and $\Delta_k$ do not change under a change in the numbering of the combinations, that is, under a permutation of the combinations in the sequence $s_1, s_2, \ldots, s_{n-k}$.

(2) $\Delta_k = \Delta_{n-k}$. This is a generalization of the assertion of problem 242.

(3) $\Delta_k \Delta_{k} = D^{\binom{n}{k}}$; (4) $\Delta_k = D^{\binom{n-1}{k}}$; (5) $\Delta_k = D^{\binom{n-1}{k}}$.

*552. Compute the determinant $P_n = |P_{ij}|$ in which $j$ divides $i$, and $P_{ij} = 0$ if $i$ does not divide $j$. Find the value of the determinant $Q_n = |Q_{ij}|$ in which $Q_{ij}$ is equal to the number of common divisors of the numbers $i$ and $j$.

*553. Euler's function is the function $\varphi(n)$ equal to the number of integers in the sequence $1, 2, \ldots, n$ that are prime to $n$. Using the preceding problem and the Gauss theorem which states that $n = \sum \varphi(s)$, where the sum is taken over all divisors $s$ of the number $n$ including 1 and $n$ itself, show that the $k$th order determinant $D = |a_{ij}|$, where $a_{ij}$ is the largest common divisor of the numbers $i$ and $j$, is equal to $\alpha_{11} \varphi(2) \varphi(3) \cdots \varphi(n)$.
560. \[2x - y - 6z + 3t + 1 = 0,\]
\[7x - 5y + 2z - 15t + 32 = 0,\]
\[x - 2y - 4z + 9t - 5 = 0,\]
\[x - y + 2z - 6t + 8 = 0.\]

561. \[2x + y + 4z + 8t = -4,\]
\[x + 3y - 6z + 2t = 3,\]
\[3x - 2y + 2z - 2t = 8,\]
\[2x - y + 2z = 4.\]

562. \[2x + y + 3z = 9,\]
\[3x - 5y + z = -4,\]
\[3x - 7y + 3z - t = -1,\]
\[4x - 7y + z = 5; \]
\[5x - 9y + 6z + 2t = 7,\]
\[4x - 6y + 3z + t = 8.\]

*564. Two systems of linear equations with the same unknowns (though not necessarily having the same number of equations) are said to be equivalent if any solution of one system satisfies the other system. (Any two systems with the same unknowns, each system having no solution, are also regarded as equivalent.)

Show that any one of the following transformations of a system of linear equations carries a given system into an equivalent system:

(a) Interchanging two equations;

(b) Multiplying both sides of one of the equations by any nonzero number;

(c) Termwise subtraction of one equation multiplied by any number from another.

Does a change in the numbering of the unknowns carry a given system into an equivalent system? Is it permissible to alter the numbering of the unknowns when solving a system of equations?

563. Prove that any system of linear equations
\[\sum_{j=1}^{n} a_{ij}x_j = b_j, \quad i = 1, 2, \ldots, s \quad (1)\]

can, via transformations of type (a), (b), and (c) of the preceding problem and via a change in the numbering of the unknowns, be reduced to the form
\[\sum_{j=1}^{n} c_{ij}y_j = d_i, \quad i = 1, 2, \ldots, s \quad (2)\]

which satisfies one and only one of the following three groups of conditions:

(a) \(c_{ii} \neq 0, \quad i = 1, 2, \ldots, n; \quad c_{ij} = 0 \quad \text{for} \quad i > j \) (in particular, the coefficients of the unknowns in all equations beyond the \(n\)th—for \(s > n\)—are zero), \(d_i = 0 \quad \text{for} \quad i = n + 1, \ldots, s \) (in this case we say that the system has been reduced to triangular form);

(b) there exists an integer \(r, \quad 0 \leq r \leq n - 1 \) such that \(c_{ii} = 0, \quad i = 1, 2, \ldots, r; \quad c_{ij} = 0 \quad \text{for} \quad i > j \); \(c_{ij} = 0 \quad \text{for} \quad i > r \) and any \(j \) equal to \(1, 2, \ldots, n \); \(d_i = 0 \quad \text{for} \quad i = r + 1, r + 2, \ldots, s \);

(c) there exists an integer \(r, \quad 0 \leq r \leq n \) such that \(c_{ii} = 0 \quad \text{for} \quad i = 1, 2, \ldots, r; \quad c_{ij} = 0 \quad \text{for} \quad i > j \); \(c_{ij} = 0 \quad \text{for} \quad i = r \) and any \(j \) equal to \(1, 2, \ldots, n \). There exists an integer \(k, \quad r + 1 \leq k \leq s \) such that \(d_k \neq 0 \).

Show that if in system (2) we restore the original numbering of the unknowns, the result will be a system equivalent to the original system (1). Then show that in case (a) the system (2) (and, hence, (1) as well) has a unique solution, in case (b), the system (2) has an infinity of solutions, where in for any values of the unknowns \(y_{r+1}, \ldots, y_n\) there exists a unique system of values of the remaining unknowns \(y_1, \ldots, y_r\); in case (c) the system (2) has no solutions. This theorem justifies the method of elimination of unknowns when solving systems of linear equations.

566. Show that if the system of linear equations (1) of the preceding problem has integer coefficients, then under all transformations involved in reducing it to the form (2) fractions can be avoided so that system (2) will have integer coefficients.

Solve the following systems of equations by the elimination method:

567. \[3x_1 - 2x_2 - 5x_3 + x_4 = 3,\]
\[2x_1 - 3x_2 + x_3 + 5x_4 = -3,\]
\[x_1 + 2x_2 - 4x_3 = -3,\]
\[x_1 - x_2 - 4x_3 + 9x_4 = 22.\]

568. \[4x_1 - 3x_2 + x_3 + 5x_4 - 7 = 0,\]
\[x_1 - 2x_2 - 2x_3 - 3x_4 = -3,\]
\[3x_1 - x_2 + 2x_3 + 1 = 0,\]
\[2x_1 + 3x_2 + 2x_3 - 8x_4 + 7 = 0.\]
569. \[2x_1 - 2x_2 + x_3 + 3 = 0,\]
\[2x_1 + 3x_2 + x_3 - 3x_4 + 6 = 0,\]
\[3x_1 + 4x_2 - x_3 + 2x_4 = 0,\]
\[x_1 + 3x_3 + x_4 - x_5 - 2 = 0.\]

570. \[x_1 + x_2 - 6x_3 - 4x_4 = 6,\]
\[3x_2 - x_3 - 6x_4 - 4x_5 = 2,\]
\[2x_1 + 3x_2 + 9x_3 + 2x_4 = 6,\]
\[3x_1 + 2x_2 + 3x_3 + 8x_4 = 7.\]

571. \[2x_1 - 3x_2 + 3x_3 + 2x_4 - 3 = 0,\]
\[6x_1 + 9x_2 - 2x_3 - x_4 + 4 = 0,\]
\[10x_1 + 3x_2 - 3x_3 - 2x_4 - 3 = 0,\]
\[8x_1 + 6x_2 + x_3 + 3x_4 + 7 = 0.\]

572. \[x_1 + 2x_2 + 5x_3 + 9x_4 = 79,\]
\[3x_1 + 13x_2 + 18x_3 + 30x_4 = 263,\]
\[2x_1 + 4x_2 + 11x_3 + 16x_4 = 146,\]
\[x_1 + 9x_2 + 9x_3 + 9x_4 = 92.\]

573. \[x_1 + x_2 + x_3 + x_4 + x_5 = 15,\]
\[x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 35,\]
\[x_1 + 3x_2 + 6x_3 + 10x_4 + 15x_5 = 70,\]
\[x_1 + 4x_2 + 10x_3 + 20x_4 + 35x_5 = 126,\]
\[x_1 + 5x_2 + 15x_3 + 35x_4 + 70x_5 = 210.\]

574. \[x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 2,\]
\[2x_1 + 3x_2 + 7x_3 + 10x_4 + 13x_5 = 12,\]
\[3x_1 + 5x_2 + 11x_3 + 16x_4 + 21x_5 = 17,\]
\[2x_1 - 7x_2 + 7x_3 + 7x_4 + 2x_5 = 57,\]
\[x_1 + 4x_2 + 5x_3 + 3x_4 + 10x_5 = 7.\]

*575. \[6x_1 + 6x_2 + 5x_3 + 18x_4 + 20x_5 = 14,\]
\[10x_1 + 9x_2 + 7x_3 + 24x_4 + 30x_5 = 18,\]
\[13x_1 + 12x_2 + 13x_3 + 27x_4 + 35x_5 = 32,\]
\[8x_1 + 6x_2 + 6x_3 + 15x_4 + 20x_5 = 16,\]
\[4x_1 + 5x_2 + 4x_3 + 15x_4 + 15x_5 = 11.\]

576. \[x_1 + x_2 + 4x_3 + 4x_4 + 9x_5 + 9 = 0,\]
\[2x_1 + 2x_2 + 17x_3 + 17x_4 + 82x_5 + 146 = 0.\]

577. \[5x_1 + 2x_2 - 7x_3 + 14x_4 = 21,\]
\[5x_1 - x_2 + 8x_3 - 13x_4 + 3x_5 = 12,\]
\[10x_1 + 2x_2 + 7x_3 - 2x_4 - x_5 = 29,\]
\[15x_1 + 3x_2 + 15x_3 + 9x_4 + 7x_5 = 130,\]
\[2x_1 - x_2 - 4x_3 + 5x_4 - 7x_5 = -13.\]

578. \[x_1 + 7x_2 + 3x_3 + x_4 = 5,\]
\[x_1 + 3x_2 + 5x_3 - 2x_4 = 3,\]
\[x_1 + 5x_2 - 9x_3 + 8x_4 = 1,\]
\[5x_1 + 18x_2 + 4x_3 + 5x_4 = 12.\]

579. \[2x_1 + 3x_2 - x_3 + x_4 = 1,\]
\[8x_1 + 12x_2 - 9x_3 + 3x_4 = 3,\]
\[4x_1 + 6x_2 + 3x_3 - 2x_4 = 3,\]
\[2x_1 + 3x_2 + 9x_3 - 7x_4 = 3.\]

580. \[4x_1 - 3x_2 + 2x_3 - x_4 = 8,\]
\[3x_1 - 2x_2 + x_3 - 3x_4 = 7,\]
\[2x_1 - x_2 - 5x_3 = 6,\]
\[5x_1 - 3x_2 + x_3 - 8x_4 = 1.\]

581. \[2x_1 - x_2 + x_3 - x_4 = 3,\]
\[4x_1 - 2x_2 - 2x_3 + 3x_4 = 2,\]
\[2x_1 - x_2 + 5x_3 - 6x_4 = 1,\]
\[2x_1 - x_2 - 3x_3 + 4x_4 = 5.\]

582. Show that a polynomial of degree \( n \) is fully determined by its values for \( n + 1 \) values of the unknown. To be more precise, show that for any distinct numbers \( x_0, x_1, x_2, \ldots, x_n \) and any numbers \( y_0, y_1, \ldots, y_n \) there exists one, and only one, polynomial \( f(x) \) of degree \( \leq n \) for which
\[f(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n.\]

583. Using the preceding problem, prove the equivalence of two definitions of the equality of polynomials of one unknown (induction can be used to prove with ease a similar
assertion for polynomials of any number of unknowns) with numerical coefficients (or coefficients of any infinite field):

(1) two polynomials are said to be equal if the coefficients of each pair of terms of the same degree are equal (this is the accepted definition in algebra);

(2) two polynomials are said to be equal if they are equal as functions, that is, if the values for each value of the unknowns are equal (this is the accepted definition in analysis).

584. Show that the definitions given in the preceding problem are not equivalent for a finite field of coefficients (construct an example).

585. Find the quadratic polynomial \( f(x) \) if we know that
\[ f(1) = -1; \quad f(-1) = 9; \quad f(2) = -3. \]

586. Find a polynomial of degree three, \( f(x) \), for which
\[ f(-1) = 0, \quad f(1) = 4, \quad f(2) = 3, \quad f(3) = 16. \]

587. What does the assertion of problem 582 mean geometrically?

588. Find a parabola of degree 3 passing through the points \((0, 1), (1, -4), (2, 3), (3, 7)\), the asymptotic direction being parallel to the axis of ordinates.

589. Find a fourth-degree parabola that passes through the points \((5, 0), (-13, 2), (-10, 3), (-2, 1), (14, -1)\), the asymptotic direction being parallel to the axis of abscissas.

Solve the following systems of linear equations, using the most suitable device in each case:

590. \[-x + y + z + t = a,\]
\[-x - y + z + t = b,\]
\[x + y - z + t = c,\]
\[x + y + z - t = d.\]

591. \[-a(x + t) + b(y + z) = c,\]
\[-a'(y + t) + b'(z + x) = c',\]
\[-a''(z + t) + b''(x + y) = c'',\]
\[x + y + z + t = d,\]
where \(a \neq b, \quad a' \neq b', \quad a'' \neq b''.\)

592. \[-ax + by + cz + dt = p,\]
\[-bx + ay + dz - ct = q.\]
599. \((3 + 2a_1) x_1 + (3 + 2a_2) x_2 + \ldots + (3 + 2a_n) x_n = 3 + 2b,\)

\((1 + 3a_1 + 2a_1^2) x_1 + (1 + 3a_2 + 2a_2^2) x_2 + \ldots + (1 + 3a_n + 2a_n^2) x_n = 1 + 3b + 2b^2,\)

\(a_1 (1 + 3a_1 + 2a_1^2) x_1 + a_2 (1 + 3a_2 + 2a_2^2) x_2 + \ldots + a_n (1 + 3a_n + 2a_n^2) x_n = b (1 + 3b + 2b^2),\)

\(a_1^{n-3} (1 + 3a_1 + 2a_1^2) x_1 + a_2^{n-3} (1 + 3a_2 + 2a_2^2) x_2 + \ldots + a_n^{n-3} (1 + 3a_n + 2a_n^2) x_n = b^{n-3} (1 + 3b + 2b^2),\)

\(a_1^{n-2} (1 + 3a_1) x_1 + a_2^{n-2} (1 + 3a_2) x_2 + \ldots + a_n^{n-2} (1 + 3a_n) x_n = b^{n-2} (1 + 3b).\)

600. Expanding the function \(\frac{x}{\ln(1 + x)}\) in a power series, we obtain \(\frac{x}{\ln(1 + x)} = 1 + h_1 x + h_2 x^2 + h_3 x^3 + \ldots.\)

Show that

\[
\begin{pmatrix}
\frac{1}{2} & 1 & 0 & 0 & \ldots & 0 \\
\frac{1}{3} & \frac{1}{2} & 1 & 0 & \ldots & 0 \\
\frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n+1} & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \ldots & \frac{1}{2}
\end{pmatrix}
\]

601. It will be recalled that \(-\frac{1}{\cos x} = 1 + \frac{c_1}{2!} x^2 + \frac{c_2}{4!} x^4 + \ldots\) where \(c_1, c_2, c_3, \ldots\) are the so-called Euler numbers. Show that

\[
\begin{pmatrix}
\frac{1}{2!} & 1 & 0 & 0 & \ldots & 0 \\
\frac{1}{4!} & \frac{1}{2!} & 1 & 0 & \ldots & 0 \\
\frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \frac{1}{(2n-6)!} & \ldots & \frac{1}{2!}
\end{pmatrix}
\]

602. In the expansion \(\frac{x}{e^{x-1}} = 1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots\), \(b_{2n} = \frac{(-1)^{n-1} B_n}{(2n)!}\), where the \(B_n\) are the so-called Bernoulli numbers. Show that

\[
\begin{pmatrix}
\frac{1}{2!} & 1 & 0 & 0 & \ldots & 0 \\
\frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \ldots & 0 \\
\frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{(2n+1)!} & \frac{1}{(2n)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \ldots & \frac{1}{2!}
\end{pmatrix}
\]

Also show that

\[
\begin{pmatrix}
\frac{1}{2!} & 1 & 0 & 0 & \ldots & 0 \\
\frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \ldots & 0 \\
\frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{(2n)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-3)!} & \ldots & \frac{1}{2!}
\end{pmatrix}
\]

for \(n > 1\).

603. Show that the Bernoulli numbers \(B_n\) that were introduced in the preceding problem can be expressed by
the following nth-order determinants:

\[
\begin{vmatrix}
    1 & 0 & 0 & \cdots & 0 \\
    3 & 1 & 0 & \cdots & 0 \\
    5 & 1 & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    2n-1 & 1 & 1 & \cdots & 1 \\
    (2n+1)! & (2n-1)! & (2n-3)! & \cdots & 1 \\
    2n & 1 & 1 & \cdots & 1 \\
    (2n+2)! & (2n-1)! & (2n-3)! & \cdots & 1 \\
    \end{vmatrix}
\]

or

\[
\begin{vmatrix}
    1 & 0 & 0 & \cdots & 0 \\
    2 & 1 & 0 & \cdots & 0 \\
    3 & 1 & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    2n & 1 & 1 & \cdots & 1 \\
    (2n+2)! & (2n-1)! & (2n-3)! & \cdots & 1 \\
    2n+1 & 1 & 1 & \cdots & 1 \\
    (2n+3)! & (2n-1)! & (2n-3)! & \cdots & 1 \\
    \end{vmatrix}
\]

604. Denote by \( s_n(k) \) the sum of the \( n \) powers of the natural numbers from 1 to \( k-1 \), that is, \( s_n(k) = 1^n + 2^n + \cdots + (k-1)^n \). Set up the equality

\[ k^n = 1 + C_n^{n-1}s_{n-1}(k) + C_n^{n-2}s_{n-2}(k) + \cdots + C_n^0s_0(k) \]

and show that

\[ s_{n-1}(k) = \frac{1}{n!} \begin{vmatrix}
    k & C_n^{n-2} & C_n^{n-2} & \cdots & C_n^0 \\
    k-1 & C_n^{n-3} & C_n^{n-2} & \cdots & C_n^1 \\
    k-2 & C_n^{n-4} & C_n^{n-3} & \cdots & C_n^2 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & 0 & 0 & \cdots & 0 \\
    \end{vmatrix} \]

605. Express as a determinant the \( n \)th coefficient \( l_n \)

of the expansion \( \tan x = 1 + l_1x^2 + l_2x^4 + \cdots \)

606. Express as a determinant the \( n \)th coefficient \( f_n \)

of the expansion \( \cot x = 1 - f_1x^2 - f_2x^4 - \cdots \)

607. Express as a determinant the \( n \)th coefficient \( a_n \)

of the expansion \( e^{-x} = 1 - a_1x - a_2x^2 - a_3x^3 - \cdots \),

then from it find the value of the determinant.

Sec. 10. The Rank of a Matrix.
The Linear Dependence of Vectors and Linear Forms

Find the ranks of the following matrices by the method of bordered minors:

608. \( \begin{pmatrix}
    2 & -1 & 3 & 2 & 4 \\
    4 & 2 & 5 & 1 & 7 \\
    2 & -1 & 1 & 8 & 2 \\
    \end{pmatrix} \)

609. \( \begin{pmatrix}
    1 & 3 & 5 & -1 \\
    2 & -1 & 3 & 4 \\
    5 & 1 & -1 & 7 \\
    \end{pmatrix} \)

610. \( \begin{pmatrix}
    3 & -1 & 3 & 2 & 5 \\
    5 & -3 & 2 & 3 & 4 \\
    1 & -3 & 5 & 0 & -7 \\
    7 & -5 & 1 & 4 & 1 \\
    \end{pmatrix} \)

611. \( \begin{pmatrix}
    4 & 3 & -5 & 2 & 3 \\
    8 & 6 & -7 & 4 & 2 \\
    4 & 3 & -8 & 2 & 7 \\
    4 & 3 & 1 & 2 & 5 \\
    8 & 6 & -4 & 1 & 6 \\
    \end{pmatrix} \)

612. Find the values of \( \lambda \) for which the matrix

\( \begin{pmatrix}
    3 & 1 & 1 & 4 \\
    \lambda & 4 & 10 & 1 \\
    1 & 7 & 17 & 3 \\
    2 & 2 & 4 & 3 \\
    \end{pmatrix} \)

has the lowest rank.

What is the rank for the values of \( \lambda \) thus found, and what is it for other values of \( \lambda \)?

613. What is the rank of the matrix

\( \begin{pmatrix}
    1 & \lambda & -1 & 2 \\
    2 & -1 & \lambda & 5 \\
    1 & 10 & -6 & 1 \\
    \end{pmatrix} \)

for various values of \( \lambda \)?

614. Let \( A \) be a matrix of rank \( r \) and let \( M_k \) be a \( k \)th-order minor in the upper left-hand corner of \( A \). Prove that by interchanging rows and interchanging columns it is pos-
able to attain fulfillment of the conditions \( M_1 \neq 0, M_2 \neq 0 \), whereas all minors of order greater than \( r \) (if such exist) are equal to zero.

616. Elementary transformations (or operations) of a matrix are:

1. multiplication of a row (or column) by a scalar different from zero;
2. addition to a row (or a column) of another row (or column) multiplied by any scalar;
3. interchanging any two rows (or columns).

Prove that elementary transformations do not alter the rank of a matrix.

617. Prove that an interchange of rows (or columns) of a matrix can be obtained by performing transformations of rows and columns solely of types (1) and (2) of problem 615.

618. Prove that by using elementary transformations (indicated in problem 615) it is possible to reduce any matrix of rank \( r \) to a form where the elements \( a_{11} = b_{11} = 1 \) and all other elements are zero.

619. Prove that by applying elementary transformations to rows alone or columns alone it is possible to reduce a square matrix to triangular form where all elements on one side of the principal diagonal are zero; what is more, the rings can be obtained either above or below the principal diagonal.

Using elementary transformations, compute the ranks of the following matrices:

620. \[
\begin{bmatrix}
35 & 34 & 17 & 43 \\
35 & 94 & 13 & 42 \\
35 & 14 & 15 & 49 \\
35 & 12 & 29 & 48
\end{bmatrix}
\]

621. \[
\begin{bmatrix}
17 & 65 & 93 & 60 \\
71 & 82 & 71 & 63 \\
60 & 72 & 14 & 63 \\
14 & 62 & 10 & 63
\end{bmatrix}
\]

622. \[
\begin{bmatrix}
17 & -28 & 45 & 11 & 39 \\
24 & -37 & 61 & 13 & 50 \\
25 & -7 & 32 & -18 & -11 \\
31 & 12 & -43 & 55 \\
42 & 13 & 20 & -55 & 68
\end{bmatrix}
\]

623. Prove that if a matrix contains \( m \) rows and \( m \) columns, then any \( s \) rows form a matrix whose rank is at least \( r + s - m \).

624. Prove that adjoining a row (or a column) to a matrix either fails to alter the rank or increases it by unity. 

625. Prove that crossing out one row (or one column) of a matrix fails to alter the rank if and only if the deleted row (or column) can be expressed linearly in terms of the remaining rows (or columns).

626. The sum of two matrices with the same number of rows and the same number of columns is a matrix each element of which is equal to the sum of the corresponding elements of the given matrices, that is, \( (a_{ij} + b_{ij}) = (a_{ij} + b_{ij}) \).

Prove that the rank of the sum of two matrices does not exceed the sum of their ranks.

627. Prove that any matrix of rank \( r \) can be expressed as a sum \( r \) of matrices of rank unity, but cannot be expressed as the sum of fewer than \( r \) such matrices.

628. Prove that if the rank of a matrix \( A \) remains unchanged after adjoining to it each column of matrix \( B \) with the same number of rows, then it will remain unchanged after adjoining to \( A \) all the columns of \( B \).

629. Prove that if the rank of a matrix \( A \) is equal to \( r \), then the matrix \( A(x) \), formed at the intersection of any \( r \) linearly independent rows and \( r \) linearly independent columns of the matrix is constant.

630. Let \( A \) be a square matrix of order \( n \) \( > 1 \), and let \( A \) be the matrix of the elements of \( A \). Determine what changes will occur in the rank \( r \) of \( A \) when the rank \( r \) of matrix \( A \) is changed.

631. Prove that if the rank of a symmetric matrix \( A \) is equal to \( r \), then the principal minors in the sizes \( r \) and \( r \) are also independent.

632. Prove that if the rank of a symmetric matrix \( A \) is equal to \( r \), and there is a principal minor of order \( r \) different from zero for which all the corresponding minor minors of orders \( r + 1 \) and \( r + 2 \)
are equal to zero, then the rank of \(A\) is \(r\) (if all principal minors are zero, then we can assume the principal minor of zeroth order, \(M_0\), to be equal to unity, and the theorem holds; for \(r = n - 1\) there are no minors of order \(r + 2\), but the theorem holds because the rank of \(A\) is equal to \(n - 1\));

(2) the rank of a symmetric matrix is equal to the highest order of nonzero principal minors of that matrix.

*632. Let \(A\) be a symmetric matrix of rank \(r\) and let \(M_k\) be a minor of the \(k\)th order located in the upper left-hand corner of \(A\). (For \(k = 0\) we assume \(M_0 = 1\).) Prove that by an appropriate interchange of rows and a corresponding interchange of columns of matrix \(A\) it is possible to attain a situation in which in the sequence of minors \(M_0 = 1, M_k, M_{k+2}, \ldots, M_r\) no two adjacent ones are zero and \(M_r \neq 0\), but minors of order greater than \(r\) (if such exist) are equal to zero.

*633. Prove that the rank of a skew-symmetric matrix is determined by its principal minors, namely:

(1) If there is a nonzero principal minor of order \(r\) for which all bordering principal minors of order \(r + 2\) are zero, then the rank of the matrix is equal to \(r\); 

(2) the rank of a skew-symmetric matrix is equal to the highest order of the nonzero principal minors of the matrix.

*634. Let \(A\) be a skew-symmetric matrix of rank \(r\) and let \(M_k\) be the \(k\)th-order minor located in the upper left-hand corner of matrix \(A\) (\(M_0 = 1\)). Prove that an appropriate interchange of rows and a corresponding interchange of columns of matrix \(A\) results in the minors \(M_k, M_{k+2}, M_{k+4}, \ldots, M_r\) being nonzero and the minors \(M_1, M_3, \ldots, M_{r-1}\) and all minors of order above \(r\) (if such exist) being zero.

635. Prove that the rank of a skew-symmetric matrix is an even number.

636. Find the linear combination \(3a_1 + 5a_2 - a_3\) of vectors:

\[ a_1 = (4, 1, 3, -2), \quad a_2 = (1, 2, -3, 2), \quad a_3 = (16, 9, 1, -3). \]

637. Find the vector \(x\) from the equation

\[ a_1 + 2a_2 + 3a_3 + 4x = 0, \]

where

\[ a_1 = (5, -3, -1, 2), \quad a_2 = (2, 1, 4), \quad a_3 = (-3, 2, -5, 4). \]

638. Find the vector \(x\) from the equation

\[ 3(a_1 - x) + 2(a_2 + x) = 5(a_3 + x), \]

where

\[ a_1 = (2, 5, 1, 3), \quad a_2 = (10, 1, 5, 10), \quad a_3 = (4, 1, -1, 1). \]

Prove that the following sets of vectors are linearly dependent or linearly independent:

639. \(a_1 = (1, 2, 3), \quad a_2 = (3, 6, 7)\),

640. \(a_1 = (4, -2, 5), \quad a_2 = (6, -3, 9)\),

641. \(a_1 = (2, -3, 1), \quad a_2 = (5, 4, 3)\),

642. \(a_1 = (3, -1, 5), \quad a_2 = (3, 2, 3)\),

643. \(a_1 = (4, -5, 2, 6), \quad a_2 = (1, -4, 3)\),

644. \(a_1 = (5, 4, 3), \quad a_2 = (6, -3, 9)\),

645. Prove that the following sets of vectors are linearly independent:

646. If from among the coordinates (components) of each vector of a given set of vectors of the same number of dimensions we choose coordinates located at definite sites (which are the same for all vectors) and if we retain the order, then we obtain a second set of vectors, which will be said to be shortened relative to the first set, which will be said to be extended relative to the second set. Prove that any shortened set for a linearly dependent set of vectors is itself linearly dependent, and any extended set for a linearly independent set of vectors is itself linearly independent.

647. Prove that a set of vectors having two equal vectors is linearly dependent.

648. Prove that a set of vectors is linearly dependent if there are two vectors that differ by a scalar factor.

649. Prove that if two vectors of a set of vectors are linearly dependent, then the whole set is linearly dependent.
650. Prove that any part of a linearly independent set of vectors is itself linearly independent.

651. Given a set of vectors
\[ a_i = (a_{i1}, a_{i2}, \ldots, a_{is}) \quad (i = 1, 2, \ldots, s; s < n). \]
Prove that if \( |a_{ij}| \geq \sum |a_{ij}| \), then the given set of vectors is linearly independent.

652. Prove that if three vectors \( a_1, a_2, a_3 \) are linearly dependent and the vector \( a_3 \) is not expressible linearly in terms of the vectors \( a_1 \) and \( a_2 \), then the vectors \( a_1 \) and \( a_2 \) differ only by a numerical factor.

653. Prove that if the vectors \( a_1, a_2, \ldots, a_k \) are linearly independent and the vectors \( a_1, a_2, \ldots, a_k, b \) are linearly dependent, then \( b \) is linearly expressible in terms of the vectors \( a_1, a_2, \ldots, a_k \).

654. Use the preceding problem to prove that every vector of a given set of vectors is linearly expressible in terms of any linearly independent subset (of the given set) that contains the maximum number of vectors.

655. Prove that an ordered set of nonzero vectors \( a_1, a_2, \ldots, a_k \) is linearly independent if and only if not a single vector can be expressed linearly in terms of the preceding ones.

656. Prove that if in front of an ordered linearly independent set of vectors \( a_1, a_2, \ldots, a_k \) we adjoin some vector \( b \), then not more than one vector of the resulting set will be expressible linearly in terms of the preceding ones.

657. Prove that if the vectors \( a_1, a_2, \ldots, a_r \) are linearly independent and can be expressed linearly in terms of the vectors \( b_1, b_2, \ldots, b_r \), then \( r \leq s \).

658. The bases of a given set of vectors is a subset with the following properties:
- If the subset is linearly independent;
- Any vector of the whole set can be linearly expressed in terms of the vectors of the subset.
Prove that
- All bases of a given set of vectors contain the same number of vectors;
- The number of vectors of any basis is the maximum number of linearly independent vectors of the given set, the number is termed the rank of the original set.

(c) If a given set of vectors has rank \( r \), then any \( r \) linearly independent vectors form a basis of that set of vectors.

659. Prove that any linearly independent subset of a given set of vectors can be completed to form a basis of the set.

660. Two sets of vectors are said to be equivalent if each vector of one set can be linearly expressed in terms of the vectors of the other and vice versa. Prove that two equivalent linearly independent sets of vectors contain the same number of vectors.

661. Prove that if the vectors \( a_1, a_2, \ldots, a_k \) are linearly expressed in terms of the vectors \( b_1, b_2, \ldots, b_l \), then the rank of the first set does not exceed the rank of the second set.

662. Given the vectors:
\[ a_1 = (0, 1, 2, 0), \quad a_2 = (1, 4, 1, 3), \]
\[ a_3 = (0, 3, 0, 4), \quad a_4 = (1, 0, 5, 7, 1), \]
\[ a_5 = (0, 1, 0, 5, 0). \]
Is it possible to choose the numbers \( c_{ij} \) \((i, j = 1, 2, \ldots, 5)\) so that the vectors
\[ b_1 = c_{11}a_1 + c_{12}a_2 + c_{13}a_3 + c_{14}a_4 + c_{15}a_5, \]
\[ b_2 = c_{21}a_1 + c_{22}a_2 + c_{23}a_3 + c_{24}a_4 + c_{25}a_5, \]
\[ b_3 = c_{31}a_1 + c_{32}a_2 + c_{33}a_3 + c_{34}a_4 + c_{35}a_5, \]
\[ b_4 = c_{41}a_1 + c_{42}a_2 + c_{43}a_3 + c_{44}a_4 + c_{45}a_5, \]
\[ b_5 = c_{51}a_1 + c_{52}a_2 + c_{53}a_3 + c_{54}a_4 + c_{55}a_5 \]
are linearly independent?

663. Prove that the vector \( b \) can be linearly expressed in terms of the vectors \( a_1, a_2, \ldots, a_k \) if and only if the rank of this set of vectors remains unchanged when the vector \( b \) is adjoined to the set.

664. Prove that
- (1) two equivalent sets of vectors have the same rank;
- (2) the converse of (1) is erroneous.
However, the following assertion holds true:
- (3) if two sets of vectors have the same rank and one of the sets is linearly expressible in terms of the other, then the sets are equivalent.

Find all values of \( k \) for which the vector \( b \) can be linearly expressed in terms of the vectors \( a_1, a_2, \ldots, a_k \).
665. \(a_1 = (2, 3, 5), a_2 = (3, 7, 3), a_3 = (1, -6, 1), b = (7, -2, 3)\).
666. \(a_1 = (4, 1, 3), a_2 = (7, 2, 4), a_3 = (4, 1, 6), b = (5, 9, 3)\).
667. \(a_1 = (3, 4, 2), a_2 = (6, 8, 7), b = (9, 12, 3)\).
668. \(a_1 = (3, 2, 5), a_2 = (2, 4, 7), b = (1, 1, 3)\).
669. \(a_1 = (3, 2, 8), a_2 = (7, 3, 9), a_3 = (5, 1, 3), b = (3, 2, 5)\).

670. Explain the answers to problems 665-669 from the standpoint of the configuration of the given vectors in space.
671. Use problem 657 to prove that more than \(n\) \(n\)-dimensional vectors are always linearly dependent.
672. Find all maximum linearly independent subsets of the following sets of vectors:
\(a_1 = (4, -4, 3, 2)\), \(a_2 = (8, -2, 6, -4)\), \(a_3 = (3, 4, 5, 6)\), \(a_4 = (4, 2, 8, -4)\).
Find all bases of the following sets of vectors:
673. \(a_1 = (1, 2, 0, 0)\), \(a_2 = (1, 2, 3, 4)\), \(a_3 = (3, 6, 0, 0)\).
674. \(a_1 = (1, 2, 3, 4)\), \(a_2 = (2, 3, 4, 5)\), \(a_3 = (3, 4, 5, 6)\), \(a_4 = (4, 5, 6, 7)\).
675. \(a_1 = (2, 1, -3, 1)\), \(a_2 = (4, 2, -6, 2)\), \(a_3 = (6, 3, -9, 3)\), \(a_4 = (1, 1, 1, 1)\).
676. \(a_1 = (1, 2, 3)\), \(a_2 = (2, 3, 4)\), \(a_3 = (3, 2, 3)\), \(a_4 = (4, 3, 4)\), \(a_5 = (1, 1, 1)\).

677. In what case does a set of vectors have a unique basis?
678. How many bases does a set of \(k + 1\) vectors of rank \(k\) have that contains proportional nonzero vectors?
679. \(a_1 = (5, 2, -3, 1)\), \(a_2 = (4, 1, -2, 3)\), \(a_3 = (1, 1, -1, -2)\), \(a_4 = (3, 4, -1, 2)\), \(a_5 = (4, -1, 15, 1)\).
680. \(a_1 = (2, 1)\), \(a_2 = (4, 2)\), \(a_3 = (1, 1)\), \(a_4 = (3, 4)\), \(a_5 = (7, 2)\).

681. \(a_1 = (1, 2, 1, 4)\), \(a_2 = (2, 3, -4, 1)\), \(a_3 = (2, -5, 8, -3)\), \(a_4 = (5, 26, -3, -12)\), \(a_5 = (3, -4, 1, 2)\).

*682. Given a set of vectors \(x_1, x_2, \ldots, x_n\), all having the same number of dimensions. The basic system of linear relations of this set of vectors is a system of relations of the form
\[\sum_{j=1}^{n} \alpha_i x_j = 0 \quad (i = 1, 2, \ldots, s),\]
which has the following two properties:
(a) the system of relations is linearly independent, which means that the set of vectors
\[\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}) \quad (i = 1, 2, \ldots, s)\]
is linearly independent;
(b) any linear dependence of the vectors \(x_1, x_2, \ldots, x_n\) is a consequence of the relations of the system, that is, if \(\sum_{j=1}^{n} \alpha_j x_j = 0\), then the vector \(a = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) is a linear combination of the vectors \(a_i, a_2, \ldots, a_n\). Prove that
(1) if \(x_1, x_2, \ldots, x_r\) is a basis of the given set of vectors and \(x_i = \sum_{j=1}^{r} \lambda_i x_j, \quad i = r + 1, r + 2, \ldots, n\), then one of the basic systems of linear relations of the given set of vectors is the system of relations \(\lambda_i = \sum_{j=1}^{n} \lambda_{ij} x_j, \quad (i = r + 1, r + 2, \ldots, n);\)
(2) all basic systems of linear relations contain the same number of relations.
(3) If some basic system of linear relations contains \( s \) relations, then any system of \( s \) linearly independent linear relations of the same set of vectors is also a basic system of linear relations.

(4) If the system of relations \( \sum_{i=1}^{n} a_{i,j} x_j = 0 \) \((i = 1, 2, \ldots, s)\) is a basic system of linear relations, then the system of relations \( \sum_{i=1}^{n} b_i x_j = 0 \) \((i = 1, 2, \ldots, s)\) will be a basic system of linear relations if and only if, assuming \( a_{i,j} (a_1,j, a_2,j, \ldots, a_n,j), \quad i = 1, 2, \ldots, s \), we have

\[
b_i = \sum_{j=1}^{n} \gamma_{i,j} a_j (i = 1, 2, \ldots, s),
\]

where the coefficients \( \gamma_{i,j} \) form a nonzero determinant of order \( s \).

Using multiplication of matrices, we can write the last \( s \) vector equations as a single matrix equation:

\[
\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0
\]

where \( A \) is a nonsingular matrix of order \( s \).

Define the basic system of linear relations for a set of linear forms in the manner of problem 682 for a set of vectors, and then find a basic system of linear relations for the following systems of linear forms:

**Problem 683**

\[
f_1 = 5x_1 + 3x_2 = 2x_3 + 4x_4, \\
f_2 = 2x_1 + 3x_2 = 3x_3 + 5x_4, \\
f_3 = 3x_1 + 5x_2 = 7x_3 + 9x_4.
\]

**Problem 684**

\[
f_1 = 8x_1 + 7x_2 = 4x_3 + 5x_4, \\
f_2 = 3x_1 + 5x_2 = 9x_3 + 4x_4, \\
f_3 = 3x_1 + 8x_2 = 2x_3 + 3x_4, \\
f_4 = 2x_1 + 3x_2 = x_3 + 7x_4, \\
f_5 = 3x_1 + 4x_2 = 17x_4.
\]

**Problem 685**

\[
f_1 = 5x_1 + 3x_2 = 2x_3 + x_4, \\
f_2 = 2x_1 + 3x_2 = 3x_3 + 4x_4, \\
f_3 = 3x_1 + 5x_2 = 3x_3 + 5x_4.
\]

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**Sec. 11. Systems of Linear Equations**

Investigate the following systems of equations for consistency and find the general solution and one particular solution in each system:

**Problem 686**

\[
f_1 = 2x_1 - x_2 + 4x_3 - 3x_4 + 5x_5, \\
f_2 = x_1 - 2x_2 + 7x_3 - 8x_4, \\
f_3 = 3x_1 - 4x_2 + x_3 - 2x_4, \\
f_4 = 4x_1 - 5x_2 + 6x_3 - 7x_4, \\
f_5 = 6x_1 - 7x_3 - 8x_4.
\]

**Problem 687**

\[
f_1 = 3x_1 + 2x_2 - 2x_3 - x_4 + 4x_5, \\
f_2 = 6x_1 - 4x_2 - 4x_3 - 2x_4 + 5x_5, \\
f_3 = 7x_1 + 5x_2 - 3x_3 - 2x_4 + 6x_5, \\
f_4 = 4x_1 + 4x_2 - 4x_3 - 3x_4 + 7x_5, \\
f_5 = 8x_1 - 7x_2 - 5x_3 - 4x_4 - 2x_5.
\]

**Problem 688**

Given a system of linear forms:

\[
f_j = \sum_{k=1}^{n} a_{j,k} x_k \quad (j = 1, 2, \ldots, s) \quad (1)
\]

and a second system of linear forms that are linearly dependent on the forms of the first system:

\[
f_j = \sum_{j=1}^{n} a_{j,j} x_k \quad (i = 1, 2, \ldots, s) \quad (2)
\]

Prove that the rank of the system of forms (2) does not exceed the rank of (1). If \( s = r \) and the determinant \( | a_{j,k} | \) is nonzero, then the ranks of both systems of linear forms coincide.
691. \[3x_1 + 4x_2 + x_4 - 2x_6 = 3,\]
\[6x_1 + 8x_2 + 2x_3 + 5x_4 = 7,\]
\[9x_1 + 12x_2 + 3x_3 + 10x_4 = 13.\]

692. \[3x_1 - 5x_2 + 2x_4 + 4x_6 = 2,\]
\[7x_1 - 4x_2 + x_3 + 3x_4 = 5,\]
\[5x_1 + 7x_2 - 4x_3 - 6x_4 = 3.\]

693. \[2x_1 + 5x_2 - 8x_3 = 8,\]
\[4x_1 + 3x_2 - 9x_3 = 9,\]
\[2x_1 + 3x_2 - 5x_3 = 7,\]
\[x_1 + 8x_2 - 7x_3 = 12.\]

694. \[3x_1 - 2x_2 + 5x_3 + 4x_4 = 2,\]
\[6x_1 - 4x_2 + 4x_3 + 3x_4 = 3,\]
\[9x_1 - 6x_2 + 3x_3 + 2x_4 = 4.\]

695. \[2x_1 - x_2 + 3x_3 - 7x_4 = 5,\]
\[6x_1 - 3x_2 + x_3 - 4x_4 = 7,\]
\[4x_1 - 2x_2 + 14x_3 - 31x_4 = 18.\]

696. \[9x_1 - 3x_2 + 5x_3 + 6x_4 = 4,\]
\[6x_1 - 2x_2 + 3x_3 + x_4 = 5,\]
\[3x_1 - x_2 + 3x_3 + 14x_4 = -8.\]

697. \[3x_1 + 2x_2 + 2x_3 + 2x_4 = 2,\]
\[2x_1 + 3x_2 + 2x_3 + 5x_4 = 3,\]
\[9x_1 + x_2 + 4x_3 - 5x_4 = 1,\]
\[2x_1 + 2x_2 + 3x_3 + 3x_4 = 5,\]
\[7x_1 + x_2 + 6x_3 - x_4 = 7.\]

698. \[x_1 + x_2 + 3x_3 - 2x_4 + 3x_5 = 1,\]
\[2x_1 + 2x_2 + 4x_3 - x_4 + 3x_5 = 2,\]
\[3x_1 + 3x_2 + 5x_3 - 2x_4 + 3x_5 = 1,\]
\[2x_1 + 2x_2 + 8x_3 - 3x_4 + 9x_5 = 2.\]

699. \[2x_1 - x_2 + x_3 + 2x_4 + 3x_5 = 2,\]
\[6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 = 3,\]
\[6x_1 - 3x_2 + 4x_3 + 6x_4 + 13x_5 = 9,\]
\[4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 1.\]

700. \[6x_1 + 4x_2 + 5x_3 + 2x_4 + 3x_5 = 1,\]
\[3x_1 + 2x_2 + 4x_3 + x_4 + 2x_5 = 3,\]
\[3x_1 + 2x_2 - 2x_3 + x_4 = 2,\]
\[9x_1 + 6x_2 + x_3 + 3x_4 + 2x_5 = -7.\]

701. \[x_1 + 2x_2 + 3x_3 - 2x_4 + x_5 = 4,\]
\[3x_1 + 6x_2 + 5x_3 - 4x_4 + 3x_5 = 5,\]
\[x_1 + 2x_2 + 7x_3 - 4x_4 + x_5 = 11,\]
\[2x_1 + 4x_2 + 2x_3 - 3x_4 + 3x_5 = 6.\]

702. \[6x_1 + 3x_2 + 2x_3 + 3x_4 + 4x_5 = 5,\]
\[4x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 = 4,\]
\[4x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 = 0,\]
\[2x_1 + x_2 + 7x_3 + 3x_4 + 2x_5 = 1.\]

703. \[8x_1 + 6x_2 + 5x_3 + 2x_4 = 21,\]
\[3x_1 + 3x_2 + 2x_3 + x_4 = 10,\]
\[4x_1 + 2x_2 + 3x_3 + x_4 = 8,\]
\[3x_1 + 5x_2 + x_3 + x_4 = 15,\]
\[7x_1 + 4x_2 + 5x_3 + 2x_4 = 18.\]

704. \[2x_1 + 3x_2 + x_3 + 2x_4 = 4,\]
\[4x_1 + 3x_2 + x_3 + x_4 = 5,\]
\[5x_1 + 11x_2 + 3x_3 + 2x_4 = 2,\]
\[2x_1 + 5x_2 + x_3 + x_4 = 1,\]
\[x_1 - 7x_2 - x_3 + 2x_4 = 7.\]

705. Prove that
(a) any system of \(s\) linear equations in \(n\) unknowns, whose matrix of the coefficients of the unknowns has rank \(r\), can, via a change in the numbering of the equations and unknowns, be brought to the form

\[
\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, \ldots, s),
\]

which has the properties that

\[m_0 = 1, \quad m_1 \neq 0, \quad m_2 \neq 0, \ldots, m_s \neq 0,\]

where \(m_k\) is a \(k\)th-order minor in the upper left-hand corner of the matrix of coefficients of the unknowns of the system (1);
(b) the system of equations (1) having the properties (2) can—via a number of subtractions of its equations, which are multiplied by suitable numbers, from subsequent equations—be reduced to the equivalent system

\[ \sum_{j=1}^{n} c_{ij} x_j = d_i \quad (i = 1, 2, \ldots, s), \quad (3) \]

which has the following properties:

\[
\begin{align*}
&c_{ij} \neq 0 \quad \text{for} \quad i = 1, 2, \ldots, r \\
&c_{ij} = 0 \quad \text{for} \quad j < i \leq r \quad \text{and also} \\
&\quad \text{for} \quad i > r \quad \text{and} \quad j = 1, 2, \ldots, n.
\end{align*}
\]

If \( d_i = 0 \) for \( i = r + 1, r + 2, \ldots, s \), then the systems (3) and (1) are consistent, and when \( r = n \) there is a unique solution; when \( r < n \) there is an infinity of solutions.

In the latter instance, \( x_{r+1}, \ldots, x_n \) are free unknowns. Using the \( r \)th equation, we can express \( x_r \) in terms of free unknowns. If this expression is put into the \((r-1)\)th equation, we find an expression of \( x_{r-1} \) in terms of the free unknowns, and so on.

Finally, using the first equation we find an expression for \( x_1 \) in terms of the free unknowns.

The resulting expressions of \( x_1, x_2, \ldots, x_r \), in terms of the free unknowns \( x_{r+1}, \ldots, x_n \) constitute the general solution of the systems (3) and (1). This means that for arbitrary values of the free unknowns we obtain from the expressions thus found the solutions of systems (3) and (1), and any solution of these systems can be obtained in this manner for suitable values of the free unknowns.

If \( d_i \neq 0 \) for even one value of \( i > r \), then the systems (3) and (1) are not consistent.

The foregoing method of investigating and solving systems of linear equations is known as the elimination method (compare with problem 565).

Using the method of elimination given in problem 705, test for consistency and find the general solution of the following systems of equations (if the original system has integral coefficients, then fractions will be avoided in the elimination process):

\[
\begin{align*}
706. \quad x_1 &+ 2x_2 + 3x_3 + x_4 = 3, \\
2x_1 &+ 3x_2 + 5x_3 + 2x_4 = 2.
\end{align*}
\]

Investigate the following systems of equations and find the general solutions, depending on the value of the parameter \( \lambda \):

\[
\begin{align*}
712. \quad 5x_1 &- 3x_2 + 2x_3 + 3x_4 = 3, \\
4x_1 &- 2x_2 + 3x_3 + 7x_4 = 1, \\
8x_1 &- 6x_2 + x_3 + 5x_4 = 9, \\
7x_1 &- 3x_2 + 7x_3 + 17x_4 = \lambda.
\end{align*}
\]
721. $x + y + z = 1$,  
722. $ax + by + cz = d$,  
723. $x + y + z = a$,  
$x + by + z = b$,  
$x + y + cz = c$.  

In what case are zero values of some of the unknowns possible?

Find the general solution and the fundamental system (or set) of solutions for the following systems of equations:

724.  
\[ x_1 + 2x_2 + 4x_3 - 3x_4 = 0, \]
\[ 3x_1 + 5x_2 + 6x_3 - 4x_4 = 0, \]
\[ 4x_1 + 5x_2 - 2x_3 + 3x_4 = 0, \]
\[ 3x_1 + 8x_2 + 24x_3 - 19x_4 = 0. \]

725.  
\[ 2x_1 - 4x_2 + 5x_3 + 3x_4 = 0, \]
\[ 3x_1 - 6x_2 + 4x_3 + 2x_4 = 0, \]
\[ 4x_1 - 3x_2 + 17x_3^2 + 11x_4 = 0. \]

726.  
\[ 3x_1 + 2x_2 + x_3 + 3x_4 + 5x_5 = 0, \]
\[ 5x_1 + 4x_2 + 3x_3 + 5x_4 + 7x_5 = 0, \]
\[ 9x_1 + 6x_2 + 3x_3 + 7x_4 + 9x_5 = 0, \]
\[ 3x_1 + 2x_2 + 4x_3 + 8x_5 = 0. \]

727.  
\[ 3x_1 + 5x_2 + 2x_3 = 0, \]
\[ 4x_1 + 7x_2 + 5x_3 = 0, \]
\[ x_1 + x_2 + 4x_3 = 0, \]
\[ 2x_1 + 9x_2 + 6x_3 = 0. \]

728.  
\[ 6x_1 - 2x_2 + 2x_3 + 5x_4 + 7x_5 = 0, \]
\[ 9x_1 - 3x_2 + 4x_3 + 8x_4 + 9x_5 = 0, \]
\[ 6x_1 - 2x_2 + 6x_3 + 7x_4 + x_5 = 0, \]
\[ 3x_1 - x_2 + 4x_3 + 4x_4 - x_5 = 0. \]

729.  
\[ x_1 - x_3 = 0, \]
\[ x_2 - x_4 = 0, \]
\[ -x_1 + x_3 - x_5 = 0, \]
\[ x_1 - x_2 + x_4 - x_5 = 0. \]
unknown is given by a homogeneous linear expression of
the parameters with integral coefficients:

735. \(2x_1 + x_2 - 4x_3 = 0\),
\(3x_1 + 5x_2 - 7x_3 = 0\),
\(4x_1 - 5x_2 - 6x_3 = 0\).

736. \(2x_1 - x_2 + 5x_3 + 7x_4 = 0\),
\(4x_1 - 2x_2 + 7x_3 + 5x_4 = 0\),
\(2x_1 - x_2 - 3x_3 - 5x_4 = 0\).

737. \(3x_1 + 2x_2 + 5x_3 + 2x_4 + 7x_5 = 0\),
\(6x_1 + 4x_2 + 7x_3 + 4x_4 + 5x_5 = 0\),
\(3x_1 + 2x_2 - x_3 + 2x_4 - 11x_5 = 0\),
\(6x_1 + 4x_2 + x_3 + 4x_4 - 13x_5 = 0\).

738. \(6x_1 - 2x_2 + 3x_3 + 4x_4 + 9x_5 = 0\),
\(3x_1 - x_2 + 2x_3 + 6x_4 + 3x_5 = 0\),
\(6x_1 - 2x_2 + 5x_3 + 20x_4 + 3x_5 = 0\),
\(9x_1 - 3x_2 + 4x_3 + 2x_4 + 15x_5 = 0\).

739. \(2x_1 + 7x_2 + 4x_3 + 5x_4 + 8x_5 = 0\),
\(4x_1 + 4x_2 + 9x_3 + 5x_4 + 4x_5 = 0\),
\(x_1 - 9x_3 - 3x_3 - 5x_4 - 14x_5 = 0\),
\(3x_1 + 5x_2 + 7x_3 + 5x_4 + 6x_5 = 0\).

740. \(3x_1 + 4x_2 + 3x_3 + 9x_4 + 6x_5 = 0\),
\(9x_1 + 8x_2 + 5x_3 + 6x_4 + 9x_5 = 0\),
\(3x_1 + 8x_2 + 7x_3 + 30x_4 + 15x_5 = 0\),
\(6x_1 + 6x_2 + 4x_3 + 7x_4 + 5x_5 = 0\).

741. Do the rows of each of the matrices
form a fundamental set of solutions for the system of equations

\[
\begin{align*}
3x_1 + 4x_2 + 2x_3 + x_4 + 6x_5 &= 0, \\
5x_1 + 9x_2 + 7x_3 + 4x_4 + 7x_5 &= 0, \\
4x_1 + 3x_2 - x_3 - x_4 + 11x_5 &= 0, \\
x_1 + 8x_2 + 3x_3 + 5x_4 - 4x_5 &= 0
\end{align*}
\]

742. Which of the rows of the matrix

\[
\begin{pmatrix}
6 & 2 & 3 & -2 & -7 \\
5 & 3 & 7 & -6 & -4 \\
8 & 0 & 5 & 6 & -13 \\
4 & -2 & 7 & 5 & -7
\end{pmatrix}
\]

form a fundamental set of solutions for the system of equations

\[
\begin{align*}
2x_1 - 5x_2 + 3x_3 + 2x_4 + x_5 &= 0, \\
5x_1 - 8x_2 + 5x_3 + 4x_4 + 3x_5 &= 0, \\
x_1 - 7x_2 + 4x_3 + 2x_4 &= 0, \\
x_1 - x_2 + x_3 + 2x_4 + 3x_5 &= 0
\end{align*}
\]

743. Prove that in place of the free unknowns we put into the general solution of a homogeneous system of linear equations of rank \( r \) having \( n \) unknowns,\(^*\) where \( r < n \), certain numbers serially taken from each row of a nonzero determinant of order \( n - r \) and if we find the appropriate values of the remaining unknowns, the result is a fundamental set of solutions, and conversely, any fundamental solution set of the given system of equations can be obtained in this manner for a suitable choice of a nonzero determinant of order \( n - r \).

744. Suppose the rows of the matrix

\[
A = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{p1} & \alpha_{p2} & \ldots & \alpha_{pn}
\end{pmatrix}
\]

form a fundamental solution set of a homogeneous system of linear equations of rank \( r \) with \( n \) unknowns \((n = r + p)\).

\[
B = \begin{pmatrix}
\beta_{11} & \beta_{12} & \ldots & \beta_{1n} \\
\beta_{21} & \beta_{22} & \ldots & \beta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{p1} & \beta_{p2} & \ldots & \beta_{pn}
\end{pmatrix}
\]

will also form a fundamental set of solutions of the same system of equations if and only if there exists a nonsingular matrix of order \( p \)

\[
C = \begin{pmatrix}
\gamma_{11} & \gamma_{12} & \ldots & \gamma_{1p} \\
\gamma_{21} & \gamma_{22} & \ldots & \gamma_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{p1} & \gamma_{p2} & \ldots & \gamma_{pp}
\end{pmatrix}
\]

such that

\[
\beta_{i,k} = \sum_{j=1}^{p} \gamma_{ij}\alpha_{jk} \quad (i = 1, 2, \ldots, p, \quad k = 1, 2, \ldots, n).
\]

Using matrix multiplication, we can write these equations as \( B = CA \).

745. Show that problem 743 is a special case of problem 744.

746. Prove that if the rank of a homogeneous system of linear equations is less by unity than the number of unknowns, then any two solutions of the system are proportional, that is, differ by only a numerical factor (which may be zero).

747. Using the theory of homogeneous systems of linear equations, solve problem 509, that is, prove that if a determinant \( D \) of order \( n \geq 4 \) is equal to zero, then the cofactors of the corresponding elements of any two rows (or columns) are proportional.\(^*\)

748. Prove that if the number of equations in a homogeneous system of linear equations is less by unity than the number of unknowns, then for a solution we can take a system of minors obtained from the matrix of coefficients by a sequential crossing out of the 1st, 2nd, etc. columns (the minors are taken with alternating signs). Furthermore, show that if this solution is not a zero solution, then any solution is obtained from it by multiplying by some scalar.\(^*\)
Use the result of the preceding problem to find a particular solution and the general solution of each of the following systems of equations:

749. \[ 5x_1 + 3x_2 + 4x_3 = 0, \]
750. \[ 4x_1 - 6x_2 + 5x_3 = 0, \]
\[ 6x_1 + 5x_2 + 2x_3 = 0. \]

751. \[ 2x_1 + 3x_2 + 5x_3 + 6x_4 = 0, \]
\[ 3x_1 + 4x_2 + 6x_3 + 7x_4 = 0, \]
\[ 3x_1 + x_2 + x_3 + 4x_4 = 0. \]

752. \[ 8x_1 - 5x_2 - 6x_3 + 3x_4 = 0, \]
\[ 4x_1 - x_2 - 3x_3 + 2x_4 = 0, \]
\[ 12x_1 - 7x_2 - 9x_3 + 5x_4 = 0. \]

753. Prove that for a system of linear equations with the number of equations one more than the number of unknowns to be consistent, it is necessary (but not sufficient) that the determinant made up of all coefficients of the unknowns and the constant terms be zero. Show that this condition will also be sufficient if the rank of the matrix of the coefficients is equal to the number of the unknowns.

754. Given: the system of linear equations

\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, \ldots, s) \]

and two solutions of this system \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) and \((\beta_1, \beta_2, \ldots, \beta_n)\) and also the number \(\lambda\). Find a system of linear equations with the same coefficients of the unknowns as in the given system and such that it has for its solution

(a) the sum of the given solutions:

\[ \alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n \]

or

(b) the product of the first of the given solutions into the number \(\lambda\):

\[ \lambda \alpha_1, \lambda \alpha_2, \ldots, \lambda \alpha_n. \]

755. Find the necessary and sufficient conditions for either the sum of two solutions or the product of one solution by a number \(\lambda = 1\) to again be a solution of the same system of linear equations.

756. Under what conditions will a given linear combination of many solutions of a given nonhomogeneous system of linear equations again be a solution of the system?
Use the theory of linear equations to solve the following problems (assume only Cartesian rectangular systems of coordinates).

764. Find the necessary and sufficient conditions for the three points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) to be collinear.

765. Write the equation of a straight line passing through the two points \((x_1, y_1), (x_2, y_2)\).

766. Find the necessary and sufficient conditions for the three straight lines

\[
a_1x + b_1y + c_1 = 0, \\
a_2x + b_2y + c_2 = 0, \\
a_3x + b_3y + c_3 = 0
\]

to be concurrent.

767. Find the necessary and sufficient conditions for the following \(n\) points of a plane to be collinear \((x_1, y_1),
(x_2, y_2), \ldots, (x_n, y_n)\).

768. Find the necessary and sufficient conditions for the following \(n\) straight lines in a plane to be concurrent:

\[
a_1x + b_1y + c_1 = 0, \\
a_2x + b_2y + c_2 = 0, \\
\ldots \ldots \ldots \ldots \ldots \ldots \\
a_nx + b_ny + c_n = 0.
\]

769. Find the necessary and sufficient conditions for four points in a plane, \((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\), not lying on a single straight line, to lie on the circumference of a circle.

770. Write down the equation of a circle passing through three points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) that do not lie on one straight line.

771. Write down the equation of a circle passing through three points \((1, 2), (4, 5), (5, 6)\) and find the centre and radius of the circle.

772. Prove that a circle passing through three points with rational coordinates has a centre in a point that also has rational coordinates.

773. Write down the equation of a second-degree curve passing through five points:

\[(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)\].
785. What system of linear equations specifies four planes in space that form a tetrahedron?

786. Give a geometrical interpretation of a system of four linear equations in three unknowns in which the ranks of all matrices made up of the coefficients of the unknowns of three equations and the rank of the augmented matrix are equal to three.

787. Consider all possible cases encountered in solving systems of linear equations involving two and three unknowns, and in each instance give a geometrical interpretation of the given system of equations.

Chapter III

Matrices and Quadratic Forms

Sec. 12. Operations Involving Matrices

Compute the following matrix products:

788. \[(3 -2) \cdot (3 4)\]

789. \[(a b) \cdot (\alpha \beta)\]

790. \[(1 -3 2) \cdot (2 5 6)\]

791. \[(5 8 -4) \cdot (3 2 5)\]

792. \[(2 -1 3 -4) \cdot (7 8 6 9)\]

793. \[(5 7 -3 -4) \cdot (1 2 3 4)\]

794. \[(2 -3) \cdot (9 -6)\]

795. \[(5 2 -2 3) \cdot (2 2 2 2)\]
796. \( \begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix} \cdot \begin{pmatrix} -28 & 33 \\ 35 & -126 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix} \)

797. \( \begin{pmatrix} 0 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 70 & 34 & -107 \\ 52 & 26 & -63 \end{pmatrix} \times \begin{pmatrix} 3 & -2 & -1 \\ 101 & 50 & -140 \end{pmatrix} = \begin{pmatrix} 27 & -48 & 10 \\ -46 & 31 & -17 \end{pmatrix} \)

798. \( \begin{pmatrix} 1 & 1 & 1 & -1 \\ -5 & -3 & -4 & 4 \\ -1 & 4 & -3 & -16 & -11 & -15 & 14 \end{pmatrix} \cdot \begin{pmatrix} 7 & 2 & 3 & 4 \\ 11 & 0 & 3 & 4 \\ 5 & 4 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 27 & -18 & 10 & 80 \end{pmatrix} \)

Evaluate the following expressions:

799. \( \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \)

800. \( \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix} \)

801. \( \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \)

802. \( \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \)

803. \( \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \)

804. \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)

805. \( \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \)

806. \( \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \)

807. \( \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \)

808. Compute \( \begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix} \) by using the equation

\( \begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ -7 & 3 \end{pmatrix} \)

809. Compute \( \begin{pmatrix} 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix} \) by using the equation

\( \begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \)

810. Prove that if for the matrices \( A \) and \( B \) both products \( AB \) and \( BA \) exist, where \( AB = BA \), then \( A \) and \( B \) are square matrices and have the same order.

811. How will the product \( AB \) of matrices \( A \) and \( B \) change if we

(a) interchange the \( i \)th and \( j \)th rows of \( A \)?
(b) add the \( j \)th row multiplied by the scalar \( c \) to the \( i \)th row of \( A \)?
(c) interchange the \( i \)th and \( j \)th columns of \( B \)?
(d) add to the \( i \)th column of \( B \) the \( j \)th column multiplied by the scalar \( c \)?

812. Using the preceding problem and the invariability of the rank under elementary transformations (see problem 615), prove that the rank of a product of two matrices does not exceed the rank of each of the factors.

813. Prove that the rank of a product of several matrices does not exceed the rank of each of the matrices being multiplied.

814. The trace of a square matrix is the sum of the elements on the principal diagonal. Prove that the trace of \( AB \) is equal to the trace of \( BA \).

815. Prove that if \( A \) and \( B \) are square matrices of the same order, \( AB \neq BA \), then

(a) \( (A + B)^2 \neq A^2 + 2AB + B^2 \).
(b) \( (A + B)(A - B) \neq A^2 - B^2 \).

816. Prove that if \( AB = BA \), then

\( (A + B)^n = A^n + nA^{n-1}B + \frac{n(n-1)}{2} A^{n-2}B^2 + \ldots + B^n \).
Here, A and B are square matrices of the same order.

817. Prove that any square matrix A may be represented uniquely in the form \( A = B + C \), where B is a symmetric matrix and C is a skew-symmetric matrix.

818. Two matrices A and B are said to commute (or to be commuting matrices) if \( AB = BA \). The square matrix A is said to be a scalar matrix if all the elements outside the principal diagonal are equal, that is, if \( A = cE \), where c is a scalar and E is the unit matrix. Prove the following assertion: for a square matrix A to be permutable (i.e., to commute with all square matrices of the same order) it is necessary and sufficient for A to be a scalar matrix.

819. A square matrix is said to be diagonal if all its elements outside the principal diagonal are zero. Prove the following assertion: for a square matrix A to commute with all diagonal matrices, it is necessary and sufficient for A to be a diagonal matrix.

820. Prove that if \( A \) is a diagonal matrix and all the elements of the principal diagonal are distinct, then any matrix that commutes with \( A \) is also a diagonal matrix.

821. Prove that premultiplication of matrix \( A \) by a diagonal matrix \( B = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \) brings about a multiplication of the rows of \( A \) by \( \lambda_1, \lambda_2, \ldots, \lambda_n \), respectively, while a postmultiplication of \( A \) by \( B \) brings about a similar change in the columns.

Find all matrices that commute with the following matrices:

822. \[ (1 \ 2) \quad 823. \quad (7 - 3) \]
     \[ (3 \ 4) \quad (5 - 2) \]

824. \[ (3 \ 1 \ 0) \quad 825. \quad (0 \ 1 \ 0) \]
     \[ (0 \ 3 \ 1) \quad (0 \ 0 \ 1) \]
     \[ (0 \ 0 \ 3) \quad (0 \ 0 \ 0) \]

826. Find all the numbers \( c \) which, when multiplied by the nonsingular matrix \( A \), do not change the determinant of the matrix.

827. Find the value of the polynomial \( f(x) = x^2 - 2x + 5 \) of the matrix \( A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \).

828. Find the value of the polynomial \( f(x) = x^3 - 7x^2 + 13x - 5 \) of the matrix \( A = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 3 & -1 \\ 2 & 2 & -1 \end{pmatrix} \).

829. Prove that the matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) satisfies the equation

\[ x^2 - (a + d) x + ad - bc = 0. \]

830. Prove that for any square matrix \( A \) there exists a polynomial \( f(x) \) that is different from zero and is such that \( f(A) = 0 \), and all such polynomials are divisible by one of them, which is uniquely defined by the condition that the leading coefficient is equal to unity (it is termed the minimal polynomial of the matrix \( A \)).

831. Prove that the equality \( AB = BA = E \) does not hold no matter what the matrices \( A \) and \( B \).

832. Find all second-order matrices whose squares are equal to the zero matrix.

833. Let \( A \) be a second-order matrix and \( k \) an integer exceeding two. Prove that \( A^k = 0 \) if and only if \( A^2 = 0 \).

834. Find all second-order matrices whose squares are equal to the unit matrix.

835. Investigate the equation \( AX = 0 \), where \( A \) is a given second-order matrix and \( X \) is the desired second-order matrix.

Find the inverses of the following matrices:

836. \[ (1 \ 2) \quad 837. \quad (3 \ 4) \]
     \[ (1 \ 4) \quad (5 \ 7) \]

838. \[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
     \[ \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \]
840. \[
\begin{pmatrix}
2 & 7 & 3 \\
6 & 3 & 4 \\
5 & -2 & -3
\end{pmatrix}
\]
841. \[
\begin{pmatrix}
3 & -4 & 5 \\
2 & -3 & 1 \\
3 & -5 & -1
\end{pmatrix}
\]
842. \[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]
843. \[
\begin{pmatrix}
1 & 2 & 2 \\
2 & 1 & -2 \\
2 & -2 & 1
\end{pmatrix}
\]
844. \[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{pmatrix}
\]
845. \[
\begin{pmatrix}
2 & 3 & 1 \\
3 & 1 & 2 \\
1 & 1 & -1 \\
1 & 1 & -1
\end{pmatrix}
\]
846. \[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\]
847. \[
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 \\
0 & 1 & 1 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]
848. \[
\begin{pmatrix}
a & a^2 & a^3 & \ldots & a^n \\
0 & a & a^2 & \ldots & a^n \\
0 & 0 & a & \ldots & a^n \\
0 & 0 & 0 & \ldots & a
\end{pmatrix}
\]
849. \[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]
850. \[
\begin{pmatrix}
1 & 2 & 3 & \ldots & n-1 & n \\
0 & 1 & 2 & \ldots & n-2 & n-1 \\
0 & 0 & 1 & \ldots & n-3 & n-2 \\
0 & 0 & 0 & \ldots & 1 & 2 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]
851. \[
\begin{pmatrix}
2 & -1 & 0 & 0 & \ldots & 0 \\
1 & 2 & -1 & 0 & \ldots & 0 \\
0 & 1 & 2 & -1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 2
\end{pmatrix}
\]
852. \[
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 0 & 1 & \ldots & 1 \\
1 & 1 & 0 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 0
\end{pmatrix}
\]
853. \[
\begin{pmatrix}
0 & 1 & 1 & \ldots & 1 \\
1 & 0 & 1 & \ldots & 1 \\
1 & 1 & 0 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 0
\end{pmatrix}
\]
854. \[
\begin{pmatrix}
1+a & 1 & 1 & \ldots & 1 \\
1 & 1+a & 1 & \ldots & 1 \\
1 & 1 & 1+a & \ldots & 1 \\
1 & 1 & 1 & \ldots & 1+a
\end{pmatrix}
\]
855. \[
\begin{pmatrix}
1+a_1 & 1 & 1 & \ldots & 1 \\
1 & 1+a_2 & 1 & \ldots & 1 \\
1 & 1 & 1+a_3 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 1+a_n
\end{pmatrix}
\]
856. Show that computing the inverse of a given $n$th-order matrix can be reduced to solving $n$ systems of linear equations, each system containing $n$ equations in $n$ unknowns and having a matrix $A$ for the matrix of coefficients of the unknowns.
Using the method of problem 856, find the inverse of the following matrices:
857. \[
\begin{pmatrix}
3 & 3 & -4 & -3 \\
0 & 6 & 1 & 1 \\
5 & 4 & 2 & 1 \\
2 & 3 & 3 & 2
\end{pmatrix}
\]
868. \[
\begin{bmatrix}
1 & 2 & 3 & \ldots & n-1 & n \\
2 & 3 & 4 & \ldots & n-2 & n-1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(n-2) & (n-3) & (n-4) & \ldots & 2 & 1 \\
(n-1) & n & \ldots & \ldots & \ldots & \ldots \\
1 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

869. \[
\begin{bmatrix}
4 & 6 \\
6 & 9 \\
\end{bmatrix} \cdot \begin{bmatrix} x \\
y \end{bmatrix} = \begin{bmatrix} 1 \\
1 \end{bmatrix}
\]

870. \[
\begin{bmatrix}
3 & -1 & 2 \\
4 & -3 & 3 \\
1 & 3 & 0 \\
\end{bmatrix} \cdot \begin{bmatrix} x \\
y \\
0 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 7 \\
1 & 1 & 7 \\
7 & 5 & 7 \\
\end{bmatrix}
\]

871. \[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
0 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
\end{bmatrix} \cdot \begin{bmatrix} x \\
y \\
\vdots \\
0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & \ldots & n \\
1 & 2 & \ldots & n-1 \\
\vdots \\
0 & 0 & \ldots & 1 \\
\end{bmatrix}
\]

872. How will the inverse matrix \( A^{-1} \) change if in the given matrix \( A \) we
(a) interchange the \( i \)th and \( j \)th rows?
(b) multiply the \( i \)th row by a nonzero scalar \( c \)?
(c) add to the \( i \)th row the \( j \)th row multiplied by a scalar \( c \), or do the same with respect to the columns?

873. An integral square matrix is said to be unimodular if its determinant is equal to \( \pm 1 \). Prove that an integral matrix has an integral inverse if and only if the given matrix is unimodular.

874. Prove that the matrix equation \( AX = B \) is solvable if and only if the rank of \( A \) is equal to the rank of the matrix \((A, B)\) obtained from \( A \) by adjoining \( B \) on the right.

875. Show that the matrix equation \( AX = 0 \), where \( A \) is a square matrix, has a nonzero solution if and only if \( |A| = 0 \).

876. Let \( A \) and \( B \) be nonsingular matrices of one and the same order. Show that the following four equalities are equivalent:

\[
AB = BA, \quad AB^{-1} = B^{-1}A, \quad A^{-1}B = BA^{-1}, \quad A^{-1}B^{-1} = B^{-1}A^{-1}.
\]

877. Let \( A \) be a square matrix and let \( f(x) \) and \( g(x) \) be any polynomials. Show that the matrices \( f(A) \) and \( g(A) \) commute, that is, \( f(A)g(A) = g(A)f(A) \).

878. Let \( A \) be a square matrix and let \( r(x) = \frac{f(x)}{g(x)} \) be a rational function of \( x \). Show that the value \( r(A) \) of the
function \( r(x) \) for \( x = A \) is defined uniquely if and only if \( g(A) \neq 0 \).

879. Find the inverse \( A^{-1} \) of the matrix \( A = \begin{pmatrix} E_k & U \\ 0 & E_l \end{pmatrix} \),
where \( E_k \) and \( E_l \) are unit matrices of orders \( k \) and \( l \) respectively, and \( U \) is an arbitrary matrix \((k, l)\) (that is, a matrix made up of \( k \) rows and \( l \) columns), while all other elements are zero.

880. A \( k \)-th skew series of order \( n \) is the term used for a square matrix \( H_k = (h_{ij}) \) of order \( n \) whose elements are defined by

\[
\begin{aligned}
h_{ij} &= 1 &\text{when } j - i &= k \\
0 &\text{when } j - i &\neq k
\end{aligned}
\]

Show that \( H_1 = H_k, H_{-1} = H_{-k} \) if \( k = 1, 2, \ldots, n - 1 \);
\( H_1 = H_{-1} = 0 \) if \( k \geq n \).

881. How does the matrix \( A \) change when pre- and post-multiplied by the matrix \( H_1 \) or \( H_{-1} \) of the preceding problem?

882. Show that the operation of transposing a matrix has the following properties:

(a) \((A + B)' = A' + B'\),
(b) \((AB)' = B'A'\),
(c) \((cA)' = cA'\),
(d) \((A^{-1})' = (A')^{-1}\),

where \( c \) is a scalar and \( A \) and \( B \) are matrices.

883. Show that if \( A \) and \( B \) are symmetric square matrices of the same order, then the matrix \( C = ABAB \ldots ABA \) is symmetric.

884. Show that

(a) the inverse of a nonsingular symmetric matrix is symmetric;
(b) the inverse of a nonsingular skew-symmetric matrix is skew-symmetric;

885. Show that for any matrix \( B \) a matrix \( A = BB' \) is symmetric.

886. Suppose \( A^* = A' \) is a matrix obtained from \( A \) by transposing and replacing all elements by complex conjugate numbers. Show that

(a) \((A + B)^* = A^* + B^*\),
(b) \((AB)^* = B^*A^*\),
(c) \((cA)^* = cA^*\),
(d) \((A^{-1})^* = (A^*)^{-1}\),

where \( c \) is a scalar and \( A \) and \( B \) are matrices that can be subjected to the indicated operations.

887. A matrix \( A \) is said to be Hermitian if \( AA^* = E \).

Show that for any matrix \( B \) with complex or real elements, the matrix \( A = B \cdot B^* \) is Hermitian.

888. Show that the product of two symmetric matrices is a symmetric matrix if and only if the given matrices commute.

889. Show that the product of two skew-symmetric matrices is a symmetric matrix if and only if the given matrices commute.

890. Prove that the product of two skew-symmetric matrices \( A \) and \( B \) is skew-symmetric if and only if \( AB = -BA \).

Give examples of skew-symmetric matrices that satisfy the condition \( AB = -BA \).

891. A square matrix \( A = (a_{ij}) \) of order \( n \) is said to be orthogonal if \( AA^* = E \), where \( E \) is the unit matrix. Show that for a square matrix \( A \) to be orthogonal it is necessary and sufficient that any one of the following conditions holds:

(a) the columns of \( A \) form an orthonormal system, that is,

\[
\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij},
\]

where \( \delta_{ij} \) is the Kronecker delta, which equals 1 for \( i = j \) and 0 when \( i \neq j \);
(b) the rows of \( A \) form an orthonormal system, that is,

\[
\sum_{k=1}^{n} a_{kj} a_{kj} = \delta_{ij},
\]

892. A square matrix \( A = (a_{ij}) \) of order \( n \) with real or complex elements is said to be unitary if \( AA^* = E \) (the meaning of \( A^* \) is the same as in problem 886). Show that \( A \) is unitary if and only if any one of the following conditions holds:

(a) \[
\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij},
\]
(b) \[
\sum_{k=1}^{n} a_{ik} a_{kj} = \delta_{ij}.
\]

\( \delta_{ij} \) is the Kronecker delta.
893. Prove that the determinant of an orthogonal matrix is equal to ±1.

894. Prove that the determinant of a unitary matrix is in modulus equal to unity.

895. Prove that if an orthogonal matrix $A$ has square submatrices $A_1, A_2, \ldots, A_n$ on the principal diagonal and zeroes on one side of the submatrices, then all elements on the other side are zero and all the matrices $A_1, A_2, \ldots, A_n$ are orthogonal.

896. Prove that for a square matrix $A$ to be orthogonal it is necessary and sufficient that its determinant be equal to ±1 and that each element be equal to its cofactor taken with its sign if $|A| = 1$ and with sign reversed if $|A| = -1$.

*897. Prove that a real square matrix $A$ of order $n \geq 3$ is orthogonal if each element is equal to its cofactor and at least one of the elements is nonzero.

*898. Prove that a real square matrix $A$ of order $n \geq 3$ is orthogonal if each element is equal to its cofactor with sign reversed and at least one of the elements is different from zero.

*899. Prove that the sum of the squares of all second-order minors lying in two rows (or columns) of an orthogonal matrix is equal to unity.

900. Prove that the sum of the square of the moduli of all second-order minors lying in two rows (or columns) of a unitary matrix is equal to unity.

901. Prove that the sum of the squares of all $k$th-order minors lying in any $k$ rows (or columns) of an orthogonal matrix is equal to unity.

902. Prove that the sum of the squares of the moduli of all $k$th-order minors lying in any $k$ rows (or columns) of a unitary matrix is equal to unity.

903. Prove that the minor of any order of an orthogonal matrix $A$ is equal to its cofactor taken with its sign if $|A| = 1$ and with sign reversed if $|A| = -1$.

904. Let $A$ be a unitary matrix and $M$ be its minor of any order, and let $M_A$ be the cofactor of $M$ in $A$. Prove that $M_A = |A| \cdot \bar{M}$, where $\bar{M}$ is the conjugate of $M$.

905. Under what conditions is a diagonal matrix orthogonal?

906. Under what conditions is a diagonal matrix unitary?

907. Verify that any one of the following three properties of a square matrix follows from the other two: reality, orthogonality, unitarity.

908. A square matrix $I$ is said to be involutory if $I^2 = E$. Show that each of the following three properties of a square matrix follows from the other two: symmetry, orthogonality, involutoriness.

909. Verify that the matrices

\[
\begin{pmatrix}
1/3 & -2/3 & -2/3 \\
-2/3 & 1/3 & -2/3 \\
-2/3 & -2/3 & 1/3
\end{pmatrix}
\]

\[
\begin{pmatrix}
1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & -1/2 & -1/2 \\
1/2 & -1/2 & 1/2 & -1/2 \\
1/2 & -1/2 & -1/2 & 1/2
\end{pmatrix}
\]

possess all three properties of the preceding problem.

910. A square matrix $P$ is said to be idempotent if $P^2 = P$. Verify that any one of the following three properties of a square matrix follows from the other two: symmetry, orthogonality, involutoriness.

911. Prove that

(a) the product of two orthogonal matrices is an orthogonal matrix;
(b) the inverse of an orthogonal matrix is orthogonal.

912. Prove that

(a) the product of two unitary matrices is a unitary matrix;
(b) the inverse of a unitary matrix is unitary.

*913. Denote by $A(i_1, i_2, \ldots, i_p)$ the minor of matrix $A$ lying at the intersection of rows labelled $i_1, i_2, \ldots, i_p$ and columns labelled $j_1, j_2, \ldots, j_p$.

Prove the validity of the following expression of the minors of the product $C = AB$ of two matrices in terms of the minors of the matrices being multiplied together:

\[
C(i_1, i_2, \ldots, i_p) = \sum \left( \begin{pmatrix} i_1, i_2, \ldots, i_p \\ k_1, k_2, \ldots, k_p \end{pmatrix} \cdot B(k_1, k_2, \ldots, k_p) \right),
\]

where $i_1 < i_2 < \ldots < i_p$. 

914. Under what conditions is a diagonal matrix orthogonal?
if \( p \) does not exceed the number of columns of matrix \( A \) or the number of rows in \( B \). Otherwise all minors of order \( p \) of matrix \( C \) are equal to zero.

914. Using the preceding problem, show that the rank of a product of two matrices does not exceed the rank of each of the factors.

915. Prove that pre- or postmultiplication of \( A \) by a nonsingular matrix does not alter the rank of the matrix.

916. The principal minor of matrix \( A \) is the minor at the intersection of rows and columns with the same number labels. Show that if the elements of matrix \( B \) are real, then all principal minors of the matrix \( A = BB' \) are nonnegative.

917. Show that for any matrix \( B \) with complex or real elements, all principal minors of matrix \( A = BB' \) are nonnegative. Here, \( B' = B^* \).

918. Show that if, using the notation of the preceding problem, \( A = BB' \), then the rank of \( A \) is equal to the rank of \( B \).

919. Prove that the sum of the principal \( k \)-th-order minors of the matrix \( AA' \) is equal to the sum of the squares of all \( k \)-th-order minors of the matrix \( A \).

920. Prove that for any square matrices \( A \) and \( B \) of order \( n \) the sum of all principal minors of a given order \( k \) \((1 \leq k \leq n)\) for matrices \( AB \) and \( BA \) is the same.

921. Let \( A \) be a real matrix of order \( n \) and let \( B \) and \( C \) be matrices made up of the first \( k \) and last \( n - k \) columns of \( A \). Prove that \( |A|^2 = |B'B| \cdot |C'C| \).

922. Let \( A = (B, C) \) be a real matrix (the meaning of the symbol \( (B, C) \) is given in problem 874). Prove that \( |A|^2 = |B'B| \cdot |C'C| \).

923. Let \( A = (a_{ij}) \) be a square real matrix of order \( n \). Prove the Hadamard inequality

\[ |A| \leq \prod_{k=1}^{n} \sum_{i=1}^{n} a_{ik}. \]

924. Prove that for any real rectangular matrix \( A = (a_{ij}) \) with \( n \) rows and \( m \) columns, the following inequality holds:

\[ |A^*A| \leq \prod_{k=1}^{m} \sum_{i=1}^{n} a_{ik}^2. \]

925. Let \( A = (B, C) \) be a matrix with complex elements. Prove that \( |A^*A| \leq |B'B| \cdot |C'C| \).

926. Let \( A = (a_{ij}) \) be a square matrix of order \( n \) with complex elements that do not exceed a number \( M \) in absolute value. Prove that the modulus of the determinant \( \det A \) does not exceed \( M^n \cdot n^n \), the estimate being exact.

927. Show that each elementary transformation of a matrix \( A \), that is, a transformation of one of the following types:

(a) transposition of two rows (or columns);
(b) multiplication of a row (or a column) by a number different from zero;
(c) adding to one row (or one column) another row (or column) multiplied by any scalar \( c \) may be obtained by multiplying the matrix \( A \) by some nonsingular matrix \( P \) on the left for transforming rows and on the right for transforming columns.

Indicate the type of such matrices.

928. A square matrix is said to be triangular if all the elements on one side of the principal diagonal are equal to zero. Show that any square matrix can be represented in the form of a product of several triangular matrices.

929. Show that any matrix \( A \) of rank \( r \) can be expressed as a product \( A = PRQ \), where \( P \) and \( Q \) are nonsingular matrices and \( R \) is a rectangular matrix of the same dimensions as \( A \) on the principal diagonal of which the first \( r \) elements are equal to unity and all the other elements are zeros.

930. Let \( A \) be a matrix of size \( m \times n \) and of rank \( r \). Let \( P = (p_{ij}) \) be a matrix of size \( s \times m \), in which \( p_{11} = \ldots = p_{2r} = \ldots = p_{r} = 1 \) and all other elements are zeros, and let \( Q = (q_{ij}) \) be a matrix of size \( n \times t \) in which \( q_{11} = \ldots = q_{2r} = \ldots = q_{r} = 1 \) and all other elements are zeros. Prove the following inequalities:

(a) the rank of \( PA \geq k + r - m \);
(b) the rank of \( AQ \geq l + r - n \);
(c) the rank of \( PAQ \geq k + l + r - m - n \).

931. Denote the rank of the matrix \( A \) by \( r_A \). Prove that for the rank of the product \( AB \) of two square matrices \( A \) and \( B \) of order \( n \) we have the following inequality:

\[ r_A + r_B - n \leq r_{AB} \leq r_A, \quad r_B \] (Sylvester's inequality)
932. Show that for the rank of the product \( AB \) of rectangular matrices \( A \) and \( B \) the Sylvester inequality of the preceding problem holds provided that \( r \) denotes the number of columns of \( A \) and the number of rows in \( B \).

933. Show that it is possible, via elementary transformations of rows alone (or columns alone), to reduce any nonsingular matrix \( A \) to the unit matrix \( E \). If the elementary operations performed on \( A \) are applied in the same order (sequence) to the unit matrix \( E \), then we obtain \( A^{-1} \), the inverse of \( A \).

Using the device of the preceding problem, find the inverse matrices of the following matrices (for convenience of computation, on the right adjoin to the given matrix \( A \) a unit matrix and perform elementary transformations of rows (which reduce \( A \) to \( E \)) on the rows of the entire matrix):

\[
\begin{pmatrix}
1 & 2 & -1 & -2 \\
3 & 8 & 0 & -4 \\
2 & 2 & -4 & -3 \\
3 & 8 & -1 & -6
\end{pmatrix}
\]

934.  
\[
\begin{pmatrix}
0 & 1 & 3 \\
2 & 3 & 5 \\
3 & 5 & 7
\end{pmatrix}
\]

936.  
\[
\begin{pmatrix}
0 & 0 & 1 & -1 \\
0 & 3 & 1 & 4 \\
2 & 7 & 6 & -1 \\
1 & 2 & 2 & -1
\end{pmatrix}
\]

937. Using the method of problem 933, find the inverses of the matrices of problems 844, 846, 847, 848, 849, 850.

938. Prove the following statement:

For the matrix \( A \) made up of \( m \) rows and \( n \) columns to have a rank of unity, it is necessary and sufficient for \( A \) to be represented as \( A = BC \), where \( B \) is a nonzero column of length \( m \) and \( C \) is a nonzero row of length \( n \).

939. Prove the following assertion:

For the matrix \( A \) consisting of \( m \) rows and \( n \) columns to have rank \( r \), it is necessary and sufficient that \( A \) be represented as \( A = BC \), where \( B \) is a matrix of \( m \) rows and \( r \) linearly independent columns, and \( C \) is a matrix of \( r \) linearly independent rows and \( n \) columns.

940. Show that if \( A \) and \( B \) are square matrices of order \( n \) and \( AB = 0 \), then \( r_A + r_B \leq n \); note that for any given matrix \( A \) the matrix \( B \) can be chosen so that \( r_A + r_B = k \), where \( k \) is any integer satisfying the condition \( r_A \leq k \leq n \).

941. Show that if \( A \) is a square matrix of order \( n \) for which \( A^2 = E \), then \( r_{E + A} + r_{E - A} = n \).

942. Two integer matrices are said to be equivalent if it is possible to pass from one to another via integer elementary transformations, that is, transformations of the following types:

(a) the transposition of two rows;
(b) multiplication of a row by \(-1\);
(c) the addition to one row of another row multiplied by an integer \( c \), and similar transformations for the columns.

Prove that the matrices \( A \) and \( B \) are equivalent if and only if \( B = PAQ \), where \( P \) and \( Q \) are square integer unimodular matrices.

943. A rectangular integer matrix \( A \) is said to be normal if its elements \( a_{11}, a_{22}, \ldots, a_{rr} \) are positive, \( a_{ii} \) is divisible by \( a_{i-1, i-1} \) \((i = 2, 3, \ldots, r)\), and all other elements are zero. Show that each integer matrix is equivalent to one and only one normal matrix; in other words, each class of equivalent integer matrices contains a normal matrix and there is only one such normal matrix.

944. Prove that each nonsingular integer matrix \( A \) may be represented as \( A = PR \), where \( P \) is a unimodular integer matrix and \( R \) is a triangular integer matrix whose elements on the main diagonal are positive, those below the main diagonal are zero, and those above the main diagonal are nonnegative and less than the elements of the main diagonal of the given column; also prove that that representation is unique.

945. Prove that a square matrix \( A \) of order \( n \) and rank \( r \) may be represented in the form \( A = PR \), where \( P \) is a nonsingular matrix and \( R \) is a triangular matrix in which the first \( r \) elements of the main diagonal are equal to unity, and all elements below the main diagonal and all elements of the last \( n - r \) rows are zero.

946. A square matrix is said to be an upper (or lower) triangular matrix if all the elements below (or, respectively, above) the main diagonal are zero. Show that the following operations applied to an upper (or lower) triangular matrix lead to an upper (or lower) triangular matrix: the addition of two matrices, the multiplication of a matrix by a scalar, the multiplication of two matrices, and the transition to an inverse matrix for a nonsingular matrix.
A square matrix is said to be nilpotent if when powered it is equal to zero. The smallest possible integer \( k \) for which \( A^k = 0 \) is called the exponent of nilpotency of the matrix \( A \). Show that a triangular matrix is nilpotent if and only if all elements of the main diagonal are zero and the exponent of nilpotency of the triangular matrix does not exceed the order of the matrix.

Show that the inverse matrix \( B = (b_{ij}) \) of the upper (or lower) triangular nonsingular matrix \( A = (a_{ij}) \) of order \( n \) is again an upper (or lower) triangular matrix, and the elements of the main diagonal of \( B \) are given by the equations \( b_{ii} = \frac{1}{a_{ii}} \) \( (i = 1, 2, \ldots, n) \), whereas the remaining elements are found from the following recurrence relations:

(a) for elements of the \( i \)th row of the upper triangular matrix:

\[
b_{ij} = \frac{1}{a_{jj}} \sum_{k=j+1}^{n} a_{ij} b_{kj} \quad (k = i+1, i+2, \ldots, n);
\]

(b) for elements of the \( k \)th column of the lower triangular matrix:

\[
b_{ik} = \frac{1}{a_{kk}} \sum_{j=k+1}^{n} a_{ij} b_{jk} \quad (i = k+1, k+2, \ldots, n).
\]

These formulas are convenient to use when computing matrices inverse to triangular matrices.

Let \( A \) be a square matrix of order \( n \) and rank \( r \), and let

\[
d_k = A \begin{pmatrix} 1, 2, \ldots, k \end{pmatrix} \begin{pmatrix} 1, 2, \ldots, k \end{pmatrix}^T \neq 0 \quad (k = 1, 2, \ldots, r).
\]

Prove that under these conditions the matrix \( A \) may be represented as the product

\[
A = BC,
\]

where \( B = (b_{ij}) \) is the lower and \( C = (c_{ij}) \) the upper triangular matrix (the definition of upper and lower triangular matrices is given in problem 346).

The first \( r \) diagonal elements of the matrices \( B \) and \( C \) may be given any values that satisfy the conditions

\[
b_{kk} c_{kk} = \frac{d_k}{d_{k-1}} \quad (k = 1, 2, \ldots, r). \tag{1}
\]

Specifying the first \( r \) diagonal elements of the matrices \( B \) and \( C \) uniquely defines the remaining elements of the first \( r \) columns of \( B \) and the first \( r \) rows of \( C \); these elements are given by the following formulas:

\[
b_{ik} = b_{kk} \frac{d_k}{d_{k-1}} \quad (1, 2, \ldots, k-1, k) \]

\[
c_{kl} = c_{kk} \frac{d_k}{d_{k-1}} \quad (1, 2, \ldots, k-1, l)
\]

\((i = k+1, k+2, \ldots, n; k = 1, 2, \ldots, r)\)

In the case of \( r < n \), all elements in the last \( n-r \) column of \( B \) may be put equal to zero, or in the last \( n-r \) rows of \( C \) the elements may be arbitrary or, contrariwise, the elements in the last \( n-r \) columns of \( B \) may be arbitrary, and all elements in the last \( n-r \) rows of \( C \) may be put equal to zero.

The arbitrary elements will not disrupt equation (2). They may be chosen so as to retain the triangular aspect of matrices \( B \) and \( C \).

Show that the representation (2) of the preceding problem may be found thus: choose the first \( r \) elements on the main diagonal of matrices \( B \) and \( C \) in any way so that they satisfy conditions (3), and compute the remaining elements of the first \( r \) columns of \( B \) and the first \( r \) rows of \( C \) by means of the following recurrence relations:

\[
a_{ik} = \sum_{j=1}^{k} b_{ij} c_{jk} \quad (i = k+1, k+2, \ldots, n; k = 1, 2, \ldots, r) \]

\[
c_{ik} = \sum_{j=1}^{k} b_{ij} c_{jk} \quad (k = 1, 2, \ldots, n; i = k+1, k+2, \ldots, r) \]
These formulas permit one to find initially the first column of \( B \) and the first row of \( C \) and then, knowing the 
\( k - 1 \) columns of \( B \) and the \( k - 1 \) rows of \( C \), to find the \( k \)th column of \( B \) and the \( k \)th row of \( C \).

*951. Prove that any symmetric matrix \( A = (a_{ij}) \) of 
order \( n \) and rank \( r \) that satisfies conditions (1) of problem 949 
may be represented in the form \( A = BB' \), where \( B \) is a lower 
triangular matrix whose elements in the last \( n - r \) columns 
are zero and the elements of the first \( r \) columns are given 
by the formulas

\[
b_{ik} = \frac{A_{ik}}{A_{ij}} \quad (i = k, k + 1, \ldots, n; k = 1, 2, \ldots, r).
\]

952. A matrix \( A \) is said to be partitioned (it is also 
called a block matrix) if the elements are split up into rectangular 
blocks by one or several horizontal and vertical partitions 
(lines). We will denote these blocks by \( A_{ij} \), where \( i \) is the 
number of the block row and \( j \) is the number of the block 
column. Show that the multiplication of two block matrices 
reduces to the multiplication of blocks regarded as separate 
elements if and only if the vertical partition of the first 
matrix corresponds to the horizontal partition of the second 
matrix. Namely, if \( A = (A_{ij}) \) is an \( m \times n \) matrix with 
partitioning of rows into groups of \( n_1, m_2, \ldots, m_s \) and 
of columns into groups of \( m_1, n_2, \ldots, n_t \), and \( B = (B_{ij}) \) 
is an \( p \times q \) matrix with partitioning of rows into groups of 
\( p_1, p_2, \ldots, p_s \) and of columns into groups of \( p_1, p_2, \ldots, p_t \), 
then \( AB = C = (C_{ij}) \) will also be a block matrix, 
and we also have

\[
C_{ik} = \sum_{j=1}^{s} A_{ij} B_{jk} \quad (i = 1, 2, \ldots, s; k = 1, 2, \ldots, n).
\]

Use this rule for multiplying block matrices to find the 
blocks of a product of the following matrices for the indicated 
partitioning into blocks for the factors

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 4 & -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.
\]

953. Show that to multiply two square partitioned matrices 
it is sufficient (but, as the example of the preceding problem 
shows, not necessary) for the diagonal blocks to be square 
and the orders of the appropriate diagonal blocks to be equal.

*954. Show that for block multiplication of a partitioned 
matrix into itself to be possible it is necessary and sufficient 
for all the diagonal blocks to be square matrices.

955. A square block (or partitioned) matrix \( A = (A_{ij}) \) 
is said to be a block-triangular matrix if all blocks on the 
main diagonal, that is, \( A_{11}, A_{22}, \ldots \), are square, and 
all the blocks on any one side of the main diagonal are zero. 
Show that if \( A \) and \( B \) are two block-triangular matrices 
with the same orders of the corresponding diagonal blocks 
and with zeros on any one side of the diagonal, then their 
product \( AB \) is also a block-triangular matrix with the same 
orders of the diagonal blocks and with zeros on the same 
side of the diagonal.

956. Show that a block-triangular matrix is nilpotent if 
and only if all the blocks on the main diagonal are nilpotent 
(the definition of nilpotency is given in problem 947).

957. Let \( A = (A_{ij}) \) be a block matrix, and let \( A_{ij} \) 
be a block of size \( m_i \times n_j \) \((i = 1, 2, \ldots, s; j = 1, 2, \ldots, t)\). 
Show that adding to the \( i \)th block row the \( j \)th row 
postmultiplied by a \( m_i \times n_j \) matrix \( Y \) may be obtained by 
post-multiplying \( A \) by a nonsingular square 
block matrix \( P \). In exactly the same way, adding to the 
\( i \)th block column the \( j \)th column premultiplied by a 
\( n_i \times n_j \) matrix \( Y \) may be obtained by 
pre-multiplying \( A \) by a nonsingular square block matrix \( Q \). 
Find the forms of the matrices \( B \) and \( Q \).

*958. Let \( R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be a partitioned matrix, where 
\( A \) is a nonsingular square matrix of order \( n \). Prove that the 
rank \( R \) is equal to \( n \) if and only if \( D = CA^{-1}B \).

*959. Let \( A \) be a nonsingular matrix of order \( n \) and let \( B \) 
be an \( n \times q \) matrix, and \( C \) a \( p \times n \) matrix. Prove that if 
the partitioned matrix \( R = \begin{pmatrix} A_1 B_1 \\ C \end{pmatrix} \) 
is reduced to the form \( R_1 = \begin{pmatrix} A_1 B_1 \\ 0 \end{pmatrix} \) 
via a series of elementary transformations of the rows with each transformation either involv-
ing only the first $n$ rows or the addition to some one row
with number label exceeding $n$ of one of the first $n$ rows
multiplied by a scalar, then $X = CA^{-1}B$.

960. Let $A$ be a nonsingular matrix of order $n$ and let $E$
be a unit matrix of the same order. Prove that if the par-
titioned matrix \[
\begin{pmatrix}
A & E \\
-E & 0
\end{pmatrix}
\]
is reduced to the form \[
\begin{pmatrix}
A_1 & B_1 \\
0 & X
\end{pmatrix}
\]
via the elementary transformations indicated in the pre-
ceding problem, then $X = A^{-1}$. Use this device to find the
inverse matrix of $A = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 4 \end{pmatrix}$.

961. Suppose we have the following system of equations:

\[
\begin{align*}
ax_1 + ax_2 + \ldots + anx_n &= b_1, \\
ax_2 + ax_2 + \ldots + anx_n &= b_2, \\
& \vdots \\
ax_n + ax_n + \ldots + anx_n &= b_n
\end{align*}
\]

with $A$ a nonsingular matrix of coefficients, with $B$ the
column of constant terms, and with $E$ a unit matrix of
order $n$.

Show that if the partitioned matrix \[
\begin{pmatrix}
A & B \\
-E & 0
\end{pmatrix}
\]
is reduced to the form \[
\begin{pmatrix}
A_1 & B_1 \\
0 & X
\end{pmatrix}
\]
by the transformations given in problem 959, then the column $X$ yields the solution of the
given system of equations.

Use this method to solve the following system:

\[
\begin{align*}
3x - y + 2z &= 7, \\
4x - 3y + 2z &= 4, \\
2x + y + 3z &= 13.
\end{align*}
\]

962. Let $A$ be a nonsingular matrix of order $n$, $B$ an
$n \times p$ matrix, and let $E$ be a unit matrix of order $n$. Show
that if the matrix \[
\begin{pmatrix}
A & B \\
-E & 0
\end{pmatrix}
\]
is brought to the form \[
\begin{pmatrix}
A_1 & B_1 \\
0 & X
\end{pmatrix}
\]
by the transformations given in problem 959, then the matrix $X$ yields the solution of the matrix equation $AX = B$.

963. Suppose all the pairs $(i, j)$ $(i = 1, 2, \ldots, n,\
\begin{align*}
A + B &= \begin{pmatrix} 2 & -7 \\ 1 & -4 \end{pmatrix}, \\
B &= \begin{pmatrix} 3 & -5 \\ 1 & -4 \end{pmatrix}.
\end{align*}

Use this method to solve the indicated equation if

\[
\begin{align*}
A &= \begin{pmatrix} 2 & -7 \\ 1 & -4 \end{pmatrix}, \\
B &= \begin{pmatrix} 3 & -5 \\ 1 & -4 \end{pmatrix}.
\end{align*}
\]

964. The right direct product of a square matrix $A$ of
order $m$ by a square matrix $B$ of order $n$ is a partitioned
matrix $A \times B = C = (C_{ij})$, where $C_{ij} = a_{i1}b_{1j}$ $(i, j =
1, 2, \ldots, m)$. Similarly, the left direct product of the
same matrices is a partitioned matrix $A \times B = D =
(D_{ij})$, where $D_{ij} = ab_{ij}$ $(i, j = 1, 2, \ldots, n)$.

(a) $(A + B) \times C = (A \times C) + (B \times C)$,

(b) $A \times (B + C) = (A \times B) + (A \times C)$,

(c) $(AB) \times (CD) = (A \times C) (B \times D)$.

965. Using the two preceding problems, prove that if $A$
is a matrix of order $m$ and $B$ is a matrix of order $n$, then
$|A \times B| = |A|^m \times |B|^n$ (see problem 560).

966. Let $A = (a_{ij})$ be a square matrix of order $n$. The
adjoint (or adjugate) of $A$ is the matrix $A = (a_{ij})$, where
\[ a_{ij} = A_{ij} \ (i, j = 1, 2, \ldots, n) \]. In other words, the adjoint of a matrix is obtained by taking the transpose of a matrix made up of the cofactors of the elements of \( A \).

Prove that

(a) \( A \hat{A} = \hat{A}A = |A|E \), where \( E \) is the unit matrix;
(b) \( (\hat{A}) = |A|^{n-2}A \) for \( n > 2 \), \( (\hat{A}) = A \) for \( n = 2 \).

*967. Show that \( (\hat{A} \hat{B}) = \hat{B} \cdot \hat{A} \), where \( \hat{A} \) is the adjoint matrix of \( A \) defined in the preceding problem.

*968. The matrix associated with a square matrix \( A \) of order \( n \) is a matrix \( \hat{A} = (\tilde{a}_{ij}) \), where \( \tilde{a}_{ij} \) is the minor of the element \( a_{ij} \) of matrix \( A \). Prove that

(a) \( (A \hat{B}) = \hat{A}B \);
(b) \( (\tilde{A}) = |A|^{n-2}A \) for \( n > 2 \), \( (\tilde{A}) = A \) for \( n = 2 \).

*969. Let \( A = (a_{ij}) \) be a square matrix of order \( n \) and let all combinations of the \( n \) numbers \( 1, 2, \ldots, n \) taken \( p \) at a time \( (k_1 < k_2 < \ldots < k_p) \) be numbered in some order \( \alpha_1, \alpha_2, \ldots, \alpha_{N_p} \), where \( N = C_n^p \); the \( p \)th associated matrix relative to \( A \) is the matrix \( A_p = (a_{ij};k_i) \) made up of appropriately arranged minors of the \( p \)th order of matrix \( A \); namely, \( a_{i_1,i_2; \ldots ;i_p} = A \left( \begin{array}{c} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_p \end{array} \right) \), where \( \alpha_i \) is the combination \( i_1 < i_2 < \ldots < i_p \), and \( \alpha_j \) is the combination \( j_1 < j_2 < \ldots < j_p \).

Prove that

(a) \( (A \hat{B})_p = A_p \hat{B}_p \);
(b) \( (\hat{E})_p = E_N \), where \( E_n \) and \( E_{N} \) are unit matrices of order \( n \) and order \( N \) respectively.

(c) If \( A \) is a nonsingular matrix, then \( (A^{-1})_p = (A_p)^{-1} \).

*970. Find a number of the combinations of \( n \) numbers \( 1, 2, \ldots, n \) taken \( p \) at a time such that relative to the triangular matrix \( A \) the associated matrix \( A_p \) defined in the preceding problem is also triangular with zeros on the same side of the diagonal.

*971. Using the properties of associated matrices, prove that if \( A \) is a square matrix of order \( n \), then \( |A_p| = |A|^{n-p} \) (see problem 551).

Sec. 13. Polynomial Matrices

Bring the following \( \lambda \)-matrices to normal diagonal form by means of elementary operations:

975. \[ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \]
976. \[ \begin{pmatrix} \lambda^2 - 1 & \lambda + 1 \\ 0 & \lambda \end{pmatrix} \]
977. \[ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + 5 \end{pmatrix} \]
978. \[ \begin{pmatrix} \lambda^2 - 1 & 0 \\ 0 & (\lambda - 1)^2 \end{pmatrix} \]
979. \[ \begin{pmatrix} \lambda + 1 & \lambda^2 + 1 & \lambda^2 \\ 3\lambda - 1 & 3\lambda^2 - 4 & \lambda^2 + 2\lambda \\ \lambda - 1 & \lambda^2 - 1 & \lambda \end{pmatrix} \]
980. \[ \begin{pmatrix} \lambda^2 & \lambda^2 - \lambda & 3\lambda^2 \\ \lambda^2 - \lambda & 3\lambda^2 - \lambda & \lambda^3 + 4\lambda^2 - 3\lambda \\ \lambda^2 + \lambda & \lambda^2 + \lambda & 3\lambda^2 + 3\lambda \end{pmatrix} \]
981. \[ \begin{pmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & -1 \\ 0 & 0 & \lambda - 2 \end{pmatrix} \]
The term invariant factors of a $\lambda$-matrix $A$ of order $n$ is used for the polynomials $E_1(\lambda), E_2(\lambda), \ldots, E_n(\lambda)$ on the main diagonal in the normal diagonal form of the matrix $A$. The divisors of the minors of $A$ are the polynomials $D_1(\lambda), D_2(\lambda), \ldots, D_n(\lambda)$, where $D_k(\lambda)$ is the greatest common divisor (with leading coefficient unity) of the minors of the $k$th order of $A$ if not all these minors are equal to zero, and $D_k(\lambda) = 0$ otherwise. Prove that $E_k(\lambda) \neq 0$ and $D_k(\lambda) \neq 0$ for $k = 1, 2, \ldots, r$, where $r$ is the rank of the matrix $A$, whereas $E_k(\lambda) = D_k(\lambda) = 0$ for $k = r + 1, \ldots, n$. Furthermore, show that $E_k(\lambda) = \frac{D_k(\lambda)}{D_{k-1}(\lambda)}$ ($k = 1, 2, \ldots, r; D_0 = 1$).

Using the minor divisors defined in problem 984, bring the following $\lambda$-matrices to normal diagonal form:

985. \[
\begin{pmatrix}
\lambda(\lambda - 1) & 0 & 0 \\
0 & \lambda(\lambda - 2) & 0 \\
0 & 0 & (\lambda - 1)(\lambda - 2)
\end{pmatrix}
\]

986. \[
\begin{pmatrix}
\lambda(\lambda - 1) & 0 & 0 \\
0 & \lambda(\lambda - 2) & 0 \\
0 & 0 & \lambda(\lambda - 3)
\end{pmatrix}
\]

987. \[
\begin{pmatrix}
\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) & 0 & 0 \\
0 & \lambda(\lambda - 2)(\lambda - 3) & 0 \\
0 & 0 & (\lambda - 1)(\lambda - 2)(\lambda - 3)
\end{pmatrix}
\]

988. \[
\begin{pmatrix}
a^2c & 0 & 0 & 0 \\
0 & b^2 & 0 & 0 \\
0 & 0 & abc^2 & 0 \\
0 & 0 & 0 & abcd^2
\end{pmatrix}
\]

where $a, b, c, d$ are pairwise relatively prime polynomials in $\lambda$. 

989. \[
\begin{pmatrix}
f(\lambda) & 0 \\
0 & g(\lambda)
\end{pmatrix}
\]

where $f(\lambda)$ and $g(\lambda)$ are polynomials in $\lambda$.

990. \[
\begin{pmatrix}
f(\lambda) & 0 \\
0 & h(\lambda)
\end{pmatrix}
\]

where $f, g, h$ are polynomials in $\lambda$ that are pairwise relatively prime and have leading coefficients unity.

991. \[
\begin{pmatrix}
fg & 0 \\
0 & gh
\end{pmatrix}
\]

where $f, g, h$ are polynomials in $\lambda$ with leading coefficients unity, coprime taken all together, but not necessarily pairwise coprime.

992. \[
\begin{pmatrix}
f & 0 \\
0 & gh
\end{pmatrix}
\]

993. \[
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{pmatrix}
\]

994. \[
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{pmatrix}
\]

995. \[
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix}
\]

996. \[
\begin{pmatrix}
\lambda + \alpha & \beta & 0 & 0 \\
-\beta & \lambda + \alpha & 0 & 1 \\
0 & 0 & \lambda + \alpha & \beta \\
0 & 0 & -\beta & \lambda + \alpha
\end{pmatrix}
\]
The following matrices are equivalent:

\[
\begin{pmatrix}
3\lambda^3 - 5\lambda + 2 & 0 & 3\lambda^3 - 6\lambda + 3 \\
2\lambda^2 - 3\lambda + 1 & \lambda - 1 & 2\lambda^2 - 4\lambda + 2 \\
2\lambda^2 - 2\lambda & 0 & 2\lambda^2 - 4\lambda + 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda + 1 & 1 & 1 \\
0 & \lambda & 1 \\
0 & 0 & \lambda \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

1003. A \(\lambda\)-matrix is said to be unimodular if its determinant is a polynomial of zero degree in \(\lambda\), that is to say, a nonzero constant. Find the normal diagonal form of a unimodular \(\lambda\)-matrix.

1004. Prove that the inverse of a \(\lambda\)-matrix is a \(\lambda\)-matrix if and only if the given matrix \(A\) is unimodular.

*1005. Prove the following statement: for two rectangular \(\lambda\)-matrices \(A\) and \(B\), each consisting of \(m\) rows and \(n\) columns, to be equivalent, it is necessary and sufficient that the following equation hold true: \(B = PAQ\), where \(P\) and \(Q\) are unimodular \(\lambda\)-matrices of order \(m\) and \(n\) respectively. Show that the required matrices \(P\) and \(Q\) can be found thus: after finding a series of elementary transformations that carry \(A\) into \(B\), apply all the transformations of the rows in the same order to the unit matrix \(E_m\) of order \(m\) and all the transformations of the columns in the same order to the unit matrix \(E_n\) of order \(n\).

Use the method given in problem 1005 to find, relative to the given \(\lambda\)-matrix \(A\), the unimodular matrices \(P\) and \(Q\) such that the matrix \(B = PAQ\) has a normal diagonal form (the matrices \(P\) and \(Q\) are not defined uniquely).

\[
A = \begin{pmatrix}
3\lambda^3 - 9\lambda^2 + 7\lambda + 1 & 2\lambda^2 - 6\lambda^2 + 7\lambda - 2 \\
3\lambda^3 - 9\lambda^2 + 9\lambda - 5 & 2\lambda^2 - 6\lambda^2 + 6\lambda - 1 \\
3\lambda^3 - 9\lambda^2 + 5\lambda + 5 & 2\lambda^3 - 6\lambda^2 + 8\lambda - 2
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
\lambda^3 - 3\lambda^2 + 3\lambda - 1 \\
\lambda^2 - 3\lambda^2 + 3\lambda - 1 \\
\lambda^3 - 3\lambda^2 + 3\lambda - 1
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
2\lambda^2 - \lambda - 1 & 17\lambda - 3\lambda^2 - 4 & \lambda^2 - 2\lambda + 1 \\
5\lambda^2 - 7\lambda + 3 & 5\lambda - 2\lambda^2 - 3 & \lambda^2 - 2\lambda + 1
\end{pmatrix}
\]
For given $\lambda$-matrices $A$ and $B$ find the unimodular $\lambda$-matrices $P$ and $Q$ that satisfy the equation $B = PAQ$ (the matrices $P$ and $Q$ are not defined uniquely—see problem 10.12).

1009. $A = \begin{pmatrix} 2\lambda^2 - \lambda + 1 & 3\lambda^3 - 2\lambda + 1 \\ 2\lambda^2 + \lambda - 1 & 3\lambda^2 + \lambda - 2 \end{pmatrix}$;

$B = \begin{pmatrix} 3\lambda^3 + 7\lambda + 2 & 3\lambda^2 + 4\lambda - 1 \\ 2\lambda^3 + 5\lambda + 1 & 2\lambda^3 + 3\lambda - 1 \end{pmatrix}$.

1010. $A = \begin{pmatrix} \lambda^2 - \lambda & \lambda^3 - \lambda \\ 2\lambda^2 - \lambda - 1 & 2\lambda^3 + \lambda^2 - 3\lambda \end{pmatrix}$;

$B = \begin{pmatrix} \lambda^3 - \lambda^2 + \lambda - 1 & 2\lambda^3 - \lambda^2 + \lambda - 2 \\ \lambda^2 - \lambda^2 & 2\lambda^2 - \lambda^2 - \lambda \end{pmatrix}$.

1011. $A = \begin{pmatrix} \lambda^2 + \lambda - 1 & \lambda + 1 \\ \lambda^2 + 2\lambda^2 + 2\lambda + 1 & \lambda^3 + \lambda^2 - 2\lambda - 1 \end{pmatrix}$;

$B = \begin{pmatrix} 4\lambda + 2 & 2\lambda + 2 & 2\lambda^2 - 2\lambda - 3 \\ 10\lambda + 2 & 5\lambda + 5 & 5\lambda^2 - 5\lambda - 2 \\ 4\lambda^2 - 7\lambda - 8 & 2\lambda^2 - 3\lambda - 5 & 2\lambda^3 - 7\lambda^2 + 2\lambda + 8 \end{pmatrix}$.

1012. $A = \begin{pmatrix} \lambda^3 - \lambda^2 + \lambda + 1 & 2\lambda^2 + 2\lambda & \lambda^2 + \lambda^2 \\ 2\lambda^3 - 3\lambda^2 - 3\lambda + 2 & 5\lambda^2 + 5\lambda + 2 & 2\lambda^3 + 2\lambda^2 \end{pmatrix}$;

$B = \begin{pmatrix} \lambda^2 + 2\lambda + 1 & \lambda^2 + \lambda \\ 2\lambda^3 + 3\lambda + 1 & \lambda^3 + 3\lambda^2 + 2\lambda & \lambda^3 + \lambda^2 \\ 3\lambda^2 + 5\lambda + 2 & 2\lambda^3 + 5\lambda^2 + 3\lambda & 2\lambda^3 + 2\lambda^2 \end{pmatrix}$.

Find the invariant factors of the following $\lambda$-matrices:

1015. $A = \begin{pmatrix} 3\lambda^2 + 2\lambda - 3 & 2\lambda - 1 & \lambda^3 + 2\lambda - 3 \\ 4\lambda^2 + 3\lambda - 5 & 3\lambda - 2 & \lambda^2 + 3\lambda - 4 \end{pmatrix}$;

$\begin{pmatrix} \lambda^2 + \lambda - 4 & \lambda - 2 & \lambda - 1 \end{pmatrix}$.

1016. $A = \begin{pmatrix} 3\lambda^3 - 2\lambda + 1 & 2\lambda^2 + \lambda - 1 & 3\lambda^2 + 2\lambda^2 - 2\lambda - 1 \\ 2\lambda^3 - 2\lambda & \lambda^2 - 1 & 2\lambda^3 + \lambda^2 - 2\lambda - 1 \\ 5\lambda^3 - 4\lambda + 1 & 3\lambda^2 + \lambda - 2 & 5\lambda^3 + 3\lambda^2 - 4\lambda - 2 \end{pmatrix}$;

$B = \begin{pmatrix} 2\lambda^2 - \lambda^2 + 2\lambda - 1 & 2\lambda^3 - 3\lambda^2 + 2\lambda - 3 \\ \lambda^3 + \lambda^2 + \lambda + 1 & \lambda^3 - 3\lambda^2 + \lambda - 3 \\ \lambda^3 - 2\lambda^2 + \lambda - 2 & \lambda^3 + \lambda^2 - \lambda - 1 \end{pmatrix}$;

$\begin{pmatrix} \lambda^2 + \lambda^2 - \lambda & \lambda^3 - 3\lambda^2 + \lambda - 3 \\ \lambda^3 - 2\lambda^2 + \lambda - 2 & 5\lambda^2 - 3\lambda^3 + 5\lambda - 2 \\ -\lambda^3 + \lambda^2 - \lambda - 1 & 7\lambda^3 - \lambda^2 - 7\lambda + 1 \end{pmatrix}$.
Find the elementary divisors of the following \( \lambda \)-matrices:

1021. \[
\begin{pmatrix}
\lambda^3 + 2 & \lambda^3 + 1 \\
2\lambda^3 - \lambda^2 - \lambda + 3 & 2\lambda^2 - \lambda - 2
\end{pmatrix}
\]

1022. \[
\begin{pmatrix}
\lambda^3 - 2\lambda^2 + 2\lambda - 1 & \lambda^2 - 2\lambda + 1 \\
2\lambda^3 - 2\lambda^2 + \lambda - 1 & 2\lambda^2 - 2\lambda
\end{pmatrix}
\]

1023. \[
\begin{pmatrix}
\lambda^2 + 2 & 2\lambda + 1 & \lambda^2 + 1 \\
\lambda^2 + 4\lambda + 4 & 2\lambda + 3 & \lambda^2 + 4\lambda + 3 \\
\lambda^2 - 4\lambda + 3 & 2\lambda - 1 & \lambda^2 - 4\lambda + 2
\end{pmatrix}
\]

1024. \[
\begin{pmatrix}
\lambda^2 - 2\lambda - 8 & \lambda^2 + 4\lambda + 4 \\
\lambda^4 + \lambda^3 - \lambda - 10 & 2\lambda^2 + 5\lambda + 2 \\
\lambda^4 + \lambda^3 - 2\lambda^2 - 3\lambda - 6 & \lambda^4 + 4\lambda + 4 \\
\lambda^4 + \lambda^3 - \lambda^2 + \lambda - 2 & \lambda^2 + 2 \\
\lambda^2 - 4 & \lambda^3 - 3\lambda - 10 \\
\lambda^2 + 3\lambda^2 - \lambda - 6 & \lambda^4 + \lambda^3 - 2\lambda - 12 \\
\lambda^4 + \lambda^3 - 2\lambda^2 - \lambda - 2 & \lambda^4 + \lambda^3 - 2\lambda^2 - 4\lambda - 8 \\
\lambda^2 + 2\lambda^2 - \lambda - 2 & \lambda^4 + \lambda^3 - \lambda^2 + \lambda - 2
\end{pmatrix}
\]

Find the elementary divisors of the following \( \lambda \)-matrices in the fields of rational, real, and complex numbers:

1026. \[
\begin{pmatrix}
\lambda^2 + 2 & \lambda^2 + 1 & 2\lambda^2 - 2 \\
\lambda^2 + 1 & \lambda^2 + 1 & 2\lambda^2 - 2 \\
\lambda^2 + 2 & \lambda^2 + 1 & 3\lambda^2 - 5
\end{pmatrix}
\]

1027. \[
\begin{pmatrix}
2\lambda^2 - 3 & \lambda^2 + 1 & \lambda^6 + 6\lambda^4 + \lambda^2 + 2 \\
\lambda^3 + 1 & 2\lambda^2 + 5 & 2\lambda^6 + 12\lambda^4 + 2\lambda^3 - 2\lambda \\
\lambda^2 + 4 & 2\lambda^6 + 12\lambda^4 + \lambda^2 - 30
\end{pmatrix}
\]
1028. 
\[
\begin{pmatrix}
\lambda^5 + 1 & \lambda^4 + \lambda^3 - 1 \\
2\lambda^4 + 3 & 2\lambda^3 - 2\lambda + 2 \lambda^2 - 2 \\
\lambda^2 - \lambda^4 + 2\lambda^3 - 2 & 2\lambda^4 - 6\lambda^3 + \lambda^2 + 6\lambda - 9
\end{pmatrix}
\]

Find the normal diagonal form of a square \(\lambda\)-matrix if we know the elementary divisors, the rank \(r\), and the order \(n\):
1029. \(\lambda^5 + 1, \lambda^4 + 1, (\lambda^4 + 1)^2, \lambda - 1, (\lambda - 1)^2; r = 4, n = 5\).
1030. \(\lambda + 2, (\lambda + 2)^2, (\lambda + 2)^3, \lambda - 2, (\lambda - 2)^3; r = 4, n = 4\).
1031. \(\lambda - 1, \lambda - 1, (\lambda - 1)^2, \lambda + 2, (\lambda + 2)^2; r = 4, n = 5\).

*1032. Prove that the set of elementary divisors of a diagonal \(\lambda\)-matrix is obtained via the union (with appropriate repetitions) of the sets of elementary divisors of all diagonal elements of that matrix.

*1033. Prove that the set of elementary divisors of a block diagonal \(\lambda\)-matrix is equal to the union (with appropriate repetitions) of the sets of the elementary divisors of all its diagonal blocks.

Using problem 1032 or 1033, find the normal diagonal form of the following \(\lambda\)-matrices:

1034. 
\[
\begin{pmatrix}
\lambda - 1 & 0 & 0 & 0 \\
0 & \lambda^2 - 1 & 0 & 0 \\
0 & 0 & \lambda^2 - 1 & 0 \\
0 & 0 & 0 & \lambda - 1
\end{pmatrix}
\]
1035. 
\[
\begin{pmatrix}
\lambda^2 - 4 & 0 & 0 & 0 \\
0 & \lambda^2 + 2\lambda & 0 & 0 \\
0 & 0 & \lambda^2 - 2\lambda & 0 \\
0 & 0 & 0 & \lambda^2 - 4
\end{pmatrix}
\]
1036. 
\[
\begin{pmatrix}
\lambda^3 + 6\lambda^2 + 9\lambda & 0 & 0 & 0 \\
0 & \lambda^2 + 2\lambda + 2\lambda^2 - 6\lambda & 0 & 0 \\
0 & 0 & \lambda^3 - 4\lambda + 4 & 0 \\
\lambda^4 + \lambda^2 - 6\lambda^2 & 0 & 0 & 0
\end{pmatrix}
\]

1037. 
\[
\begin{pmatrix}
0 & 0 & 0 & \lambda^4 - 2\lambda^2 - 2\lambda - 1 \\
0 & \lambda^2 + 2\lambda - 1 & 0 & 0 \\
0 & 0 & \lambda^2 + 2\lambda - 1 & 0 \\
\lambda^2 - 2\lambda - 1 & 0 & 0 & 0
\end{pmatrix}
\]
1038. 
\[
\begin{pmatrix}
\lambda^2 - 2\lambda - 3 & \lambda^2 + \lambda - 2 & 0 & 0 \\
2\lambda^3 - 2\lambda^2 - 4 & 2\lambda^2 + \lambda - 3 & 0 & 0 \\
0 & 0 & \lambda^3 + 2\lambda & \lambda^2 + 6\lambda - 2 \\
0 & 0 & \lambda^3 + \lambda - 2 & \lambda^2 + 5\lambda - 7
\end{pmatrix}
\]
1039. 
\[
\begin{pmatrix}
\lambda^2 - \lambda - 2 & \lambda^3 + \lambda^2 - \lambda - 1 & 0 & 0 \\
\lambda^3 - 4 & \lambda^3 + 2\lambda^2 - \lambda - 2 & 0 & 0 \\
0 & 0 & \lambda^2 + 2\lambda & \lambda^2 + 6\lambda - 2 \\
0 & 0 & \lambda^3 + \lambda - 2 & \lambda^2 + 5\lambda - 7
\end{pmatrix}
\]
1040. 
\[
\begin{pmatrix}
0 & 0 & \lambda^3 - \lambda^2 - \lambda - 2 & \lambda^3 - 2\lambda - 4 \\
0 & 0 & \lambda^3 + \lambda - 6 & \lambda^3 + 6\lambda - 6 \\
\lambda^3 - 2\lambda + 1 & \lambda - 2 & 0 & 0 \\
\lambda^3 - 2\lambda^2 + 6\lambda - 1 & \lambda^3 - 2\lambda + 5 & 0 & 0
\end{pmatrix}
\]
1041. 
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 - 2\lambda - 1 & 0 \\
\lambda^2 + 2\lambda - 3 & \lambda^3 + \lambda - 2 & 0 \\
\lambda^3 + 2\lambda^2 + \lambda - 4 & \lambda^2 + 2\lambda^2 - 3 & 0 \\
\lambda^2 - 2\lambda - 3 & \lambda^3 + \lambda^2 - 9\lambda & 0 \\
\lambda^2 - \lambda - 2 & \lambda^3 + 2\lambda^2 - 5\lambda & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Defining the equivalence and normal diagonal form of integral matrices as is done in problems 942 and 943, find the greatest common divisors \( D_k \) of the \( k \)th-order minors of the following matrices by means of reducing them to normal diagonal form via elementary operations:

\[
\begin{pmatrix}
0 & 2 & 4 & -1 \\
6 & 12 & 14 & 5 \\
0 & 4 & 14 & -1 \\
10 & 6 & -4 & 11
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 6 & -9 & -3 \\
12 & 24 & 9 & 9 \\
30 & 42 & 45 & 27 \\
60 & 78 & 81 & 63
\end{pmatrix}
\]

1044. Prove that by means of elementary transformations of rows alone (or columns alone) it is possible to reduce any \( \lambda \)-matrix of rank \( r \) to triangular or trapezoidal form with the zeros obtained either above or below the main diagonal, at one's pleasure; the nonzero elements will lie only in the first \( r \) rows (or the first \( r \) columns).

*1045. Prove that every nonsingular \( \lambda \)-matrix \( A \) may be expressed in the form \( A = PR \), where \( P \) is a unimodular \( \lambda \)-matrix and \( R \) is a triangular \( \lambda \)-matrix whose elements on the main diagonal have leading coefficient unity, those below the main diagonal are zero, and those above the main diagonal are of a power less than that of the element of the main diagonal of the same column (or are zero); this representation is unique.


All problems in this section are posed in matrix form. In particular, the properties of characteristic roots (eigenvalues) of a matrix and the reduction of the matrix to the Jordan form are considered without regard for the properties of the eigenvectors and the invariant subspaces of the appropriate linear transformation. This relationship is particular, finding the basis in which the matrix of the given linear transformation has the Jordan form) is considered in Chapter IV. This does not prevent using the problems of this section when studying the properties of linear transformations to the extent in which the relationship between linear transformations and their matrices in some given basis has been studied.

1046. A matrix \( A \) is said to be similar to a matrix \( B \) (this is denoted as \( A \sim B \)) if there exists a nonsingular matrix \( T \) such that \( B = T^{-1}AT \). Show that the similarity relation has the following properties:

(a) \( A \sim A \); (b) if \( A \sim B \), then \( B \sim A \);
(c) if \( A \sim B \) and \( B \sim C \), then \( A \sim C \).

1047. Prove that if at least one of the two matrices \( A \) and \( B \) is nonsingular, then the matrices \( AB \) and \( BA \) are similar.

Give an example of two singular matrices \( A, B \) for which the matrices \( AB \) and \( BA \) are not similar.

*1048. Find all matrices each of which is similar only to itself.

1049. Let a matrix \( B \) be obtained from \( A \) by interchanging the \( i \)th and \( j \)th rows and also the \( i \)th and \( j \)th columns. Prove that \( A \) and \( B \) are similar and find the nonsingular matrix \( T \) for which \( B = T^{-1}AT \).

*1050. Show that the matrix \( A \) is similar to the matrix \( B \) obtained from \( A \) by a reflection about its centre.

1051. Let \( i_1, i_2, \ldots, i_n \) be any permutation of the numbers 1, 2, \ldots, \( n \). Prove that the matrices

\[
\begin{pmatrix}
\begin{array}{cccc}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
& \cdots & & \\
\alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}
\end{array}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\begin{array}{cccc}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
& \cdots & & \\
\alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}
\end{array}
\end{pmatrix}
\]

are similar.

1052. Suppose we have two similar matrices \( A \) and \( \Sigma \). Show that the collection of all nonsingular matrices \( T \) for which \( B = T^{-1}AT \) can be obtained from the collection of all nonsingular matrices permutable with \( A \), by postmultiplying these matrices by any one matrix \( T_0 \) with the property \( B = T_0^{-1}AT_0 \).
1053. Prove that if a matrix $A$ is similar to a diagonal matrix, then the $p$th associated matrix $A_p$ (see problem 969) is also similar to the diagonal matrix.

1054. Prove that if two matrices $A$ and $B$ are similar to diagonal matrices, then their Kronecker product $A \otimes B$ (see problem 963) is also a matrix similar to the diagonal matrix.

1055. Prove that if the matrices $A$ and $B$ are similar, then the $p$th associated matrices $A_p$ and $B_p$ (under any two arrangements of the combinations of $p$ numbers of rows and columns taken $p$ at a time) are also similar.

1056. Prove that if the matrices $A, B$ are similar, respectively, to the matrices $A_1, B_1,$ then the Kronecker products $A_1 \times B_1$ (under any two arrangements of pairs of indices) are also similar.

1057. Prove that if the matrices $A, B$ are similar, then the $p$th associated matrices $A, B$ (under any two arrangements of the combinations of $p$ numbers of rows and columns taken $p$ at a time) are also similar.

1058. Take the matrix

$$A = \begin{pmatrix} -2\lambda^2 + 3\lambda + 1 & -\lambda^2 + 3\lambda + 2 & -\lambda + 6 \\ -3\lambda^2 + 2\lambda + 8 & -2\lambda^2 + 3\lambda + 1 & -\lambda + 4 \\ -\lambda^2 + 3\lambda + 1 & 2\lambda^2 + 5\lambda + 3 & -\lambda + 4 \end{pmatrix}$$

and divide it on the left by $B - \lambda E$, where $B = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$.

1059. Take the matrix

$$A = \begin{pmatrix} -\lambda^2 + \lambda^2 + 3\lambda + 6 & \lambda^2 + 2\lambda & \lambda^2 + 2\lambda + 6 \\ -2\lambda^2 + 2\lambda + 8 & \lambda^2 + 3\lambda + 5 & \lambda^2 + 2\lambda + 9 \\ -\lambda^2 + \lambda^2 + 3\lambda + 6 & \lambda^2 + 2\lambda + 8 & \lambda^2 + 3\lambda + 5 \end{pmatrix}$$

and divide it on the right by $B - \lambda E$, where $B = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

*1060. Prove that if two matrices $A$ and $B$ with numerical elements (or elements taken from any other field) are similar, then their characteristic matrices $A$ and $B - \lambda E$ are equivalent.

*1061. Prove that if the characteristic matrices $A$ and $B - \lambda E$ of two matrices $A$ and $B$ are equivalent, then the matrices themselves are similar. Also show that $B - \lambda E = P (A - \lambda E) Q$, where $P$ and $Q$ are unimodular matrices and $P_0$ and $Q_0$ are the remainders resulting from the division of $P$ on the left and of $Q$ on the right by $B - \lambda E$; that is, the matrix $Q$ accomplishes a similarity transformation of the matrix $A$ into the matrix $B$.

1062. Prove that any square matrix $A$ is similar to its transpose $A'$.

Determine whether the following matrices are similar or not:

1063. 

$$A = \begin{pmatrix} 3 & 2 & -5 \\ 7 & -10 & 0 \\ -3 & -4 & 7 \end{pmatrix} ; \quad B = \begin{pmatrix} 6 & 20 & -34 \\ 32 & -10 & -15 \\ 4 & 20 & -32 \end{pmatrix}$$

1064. 

$$A = \begin{pmatrix} 1 & 5 & -15 \\ 1 & 2 & -5 \\ 1 & 2 & -2 \end{pmatrix} ; \quad B = \begin{pmatrix} 3 & -7 & 19 \\ 1 & -7 & 11 \\ 1 & 1 & -7 \end{pmatrix}$$

1065. 

$$A = \begin{pmatrix} 4 & 6 & -15 \\ 2 & -4 & 17 \\ 1 & 2 & -5 \end{pmatrix} ; \quad B = \begin{pmatrix} 1 & -3 & 3 \\ 2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$$

1066. 

$$C = \begin{pmatrix} -13 & -70 & 119 \\ -4 & -19 & 34 \\ -4 & -20 & 35 \end{pmatrix}$$
Using the method given in problem 1061, find, for the given matrices $A$ and $B$, a nonsingular matrix $T$ such that $B = T^{-1}AT$ (the desired matrix $T$ is not defined uniquely):

*1067. $A = \begin{pmatrix} 5 & -1 \\ 9 & -1 \end{pmatrix}$; $B = \begin{pmatrix} 38 & -81 \\ 16 & -34 \end{pmatrix}$.

*1068. $A = \begin{pmatrix} 17 & -6 \\ 45 & -46 \end{pmatrix}$; $B = \begin{pmatrix} 14 & -60 \\ 3 & -13 \end{pmatrix}$.

*1069. $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 6 & 5 \end{pmatrix}$; $R = \begin{pmatrix} 24 & -11 & -22 \\ 20 & -8 & -20 \\ 12 & -6 & -10 \end{pmatrix}$.

*1070. Prove that the coefficients of the characteristic polynomial $|A - \lambda E|$ of the matrix $A$ can be expressed in terms of the elements of that matrix in the following manner:

$$|A - \lambda E| = (\lambda)^n + c_1 (\lambda)^{n-1} + c_2 (\lambda)^{n-2} + \ldots + c_n,$$

where $c_k$ is the sum of all principal minors of order $k$ of the matrix $A$ (a principal minor is a principal minor if the numbers of the rows coincide with the numbers of the columns).

*1071. Find the eigenvalues (the roots of the characteristic polynomial) of the matrix $A' A$, where $A = (a_{ij}$, $a''_{ij})$ and $A'$ is the transpose of $A$.

*1072. Prove that the sum of the eigenvalues of matrix $A$ is equal to its trace (that is, the sum of the elements of the principal diagonal), and the product of these numbers is equal to the determinant $|A|$.

1073. Prove that all eigenvalues of matrix $A$ are nonzero if and only if $A$ is nonsingular.

*1074. Let $p > 0$ be the multiplicity of the root $\lambda_0$ of the characteristic polynomial $|A - \lambda E|$ of matrix $A$ of order $n$, let $r$ be the rank and $d = n - r$ the nullity of the matrix $A - \lambda_0 E$. Prove the truth of the following inequalities:

$$1 \leq d = n - r \leq p.$$
1080. Prove that if \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of matrix \( A \) and \( f(x) \) is a polynomial, then \( f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n) \) are eigenvalues of the matrix \( f(A) \).

1081. Prove that if \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of matrix \( A \) and \( f(x) = \frac{g(x)}{h(x)} \) is a rational function defined for the value \( x = A \) (that is, the function satisfies the condition \( h(A) \neq 0 \)), then \( f(A) = f(\lambda_1) f(\lambda_2) \ldots f(\lambda_n) \) and the numbers \( f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n) \) are the eigenvalues of the matrix \( f(A) \).

1082. Prove that if \( A \) and \( B \) are square matrices of the same order, then the characteristic polynomials of the matrices \( AB \) and \( BA \) coincide.

1083. Find the eigenvalues of the cyclic matrix

\[
A = \begin{bmatrix}
1 & a_2 & a_3 & \ldots & a_n \\
0 & 1 & a_2 & \ldots & a_{n-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & a \nend{bmatrix}
\]

*1084. Find the eigenvalues of the \( n \)th-order matrix

\[
A = \begin{bmatrix}
0 & -1 & 0 & \ldots & 0 \\
1 & 0 & -1 & \ldots & 0 \\
0 & 1 & 0 & \ldots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{bmatrix}
\]

*1085. A Jordan matrix is a block-diagonal matrix with diagonal blocks of the form

\[
\alpha \ 1 \ 0 \ \cdots \ 0 \\
0 \ \alpha \ 1 \ \cdots \ 0 \\
\vdots & \vdots & \ddots & \ddots \end{bmatrix}
\]

These are the so-called Jordan submatrices. The Jordan form of the matrix \( A \) is a Jordan matrix \( A_J \) similar to the matrix \( A \).

Taking advantage of the theorem which states that the collection of elementary divisors of a block-diagonal matrix is equal to the union of the collections of elementary divisors of its diagonal blocks (see problem 1033), prove that any matrix \( A \) has, over the field of complex numbers (or over any field containing all the eigenvalues of \( A \)), a Jordan form which is unique up to the order of the blocks.

Write the Jordan form \( A_J \) of the matrix \( A \) if \( A \) have the invariant factors \( E_i(\lambda) \) \( (i = 1, 2, \ldots, n) \) of its characteristic matrix \( A - \lambda E \) :

1086. \( E_1(\lambda) = E_2(\lambda) = 1, E_3(\lambda) = E_4(\lambda) = \lambda, E_5(\lambda) = E_6(\lambda) = (\lambda - 1)(\lambda + 2) \).

1087. \( E_1(\lambda) = E_2(\lambda) = E_3(\lambda) = \lambda + 1, E_4(\lambda) = (\lambda + 1)^2, E_5(\lambda) = (\lambda - 1)^2 (\lambda - 5) \).

1088. \( E_1(\lambda) = E_2(\lambda) = 1, E_3(\lambda) = \lambda - 2, E_4(\lambda) = \lambda^2 - 4 \).

1089. Prove that for any square \( \lambda \)-matrix \( A(\lambda) \) of order \( n \), the determinant of which is a polynomial in \( \lambda \) of degree \( n \), there exists a numerical matrix \( B \) of order \( n \) such that the matrix \( A(\lambda) \) is equivalent to the characteristic matrix \( B - \lambda E \).

Find the Jordan form of each of the following matrices:

1090. \[
\begin{pmatrix}
0 & 1 & 0 \\
-4 & 4 & 0 \\
-2 & 1 & 2
\end{pmatrix}
\]

1091. \[
\begin{pmatrix}
2 & 6 & -15 \\
11 & -5 & 0 \\
12 & -6 & 0
\end{pmatrix}
\]

1092. \[
\begin{pmatrix}
9 & -6 & -2 \\
18 & -12 & -3 \\
18 & -9 & -6
\end{pmatrix}
\]

1093. \[
\begin{pmatrix}
4 & 6 & -15 \\
13 & -5 & 0 \\
12 & -4 & 0
\end{pmatrix}
\]

1094. \[
\begin{pmatrix}
0 & -4 & 0 \\
1 & -4 & 0 \\
1 & -2 & -2
\end{pmatrix}
\]

1095. \[
\begin{pmatrix}
12 & -6 & -2 \\
18 & -9 & -3 \\
18 & -9 & -3
\end{pmatrix}
\]

1096. \[
\begin{pmatrix}
4 & 5 & 2 \\
5 & -7 & 3 \\
6 & -9 & 4
\end{pmatrix}
\]

1097. \[
\begin{pmatrix}
5 & -3 & 2 \\
6 & -4 & 4 \\
4 & -4 & 5
\end{pmatrix}
\]

1098. \[
\begin{pmatrix}
1 & -3 & 3 \\
-2 & -6 & 13 \\
-1 & -4 & 8
\end{pmatrix}
\]

1099. \[
\begin{pmatrix}
7 & -12 & 6 \\
10 & -19 & 10 \\
12 & -24 & 13
\end{pmatrix}
\]
Find the Jordan form of a matrix of the form
\[
\begin{pmatrix}
I & A_{12} & A_{13} & \ldots & A_{1n} \\
0 & A_{22} & A_{23} & \ldots & A_{2n} \\
0 & 0 & A_{33} & \ldots & A_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A_{nn}
\end{pmatrix}
\]
provided that \( a_{12}, a_{23}, \ldots, a_{n-1,n} \neq 0 \).

Prove that if the characteristic polynomial \( |A - \lambda E| \) of a matrix \( A \) does not have any multiple roots, then \( A \) is similar to a diagonal matrix (the elements of the matrix \( T \) that transforms \( A \) into diagonal form belong to the field that contains all eigenvalues of \( A \)).

Prove that a matrix \( A \) over a given field \( P \) is similar to a diagonal matrix if and only if the last invariant factor \( E_n(\lambda) \) of the characteristic matrix \( A - \lambda E \) does not have any multiple roots and all its roots belong to the field \( P \).

Determine whether the following matrices are similar to diagonal matrices in the fields of rational, real, and complex numbers:

\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & n \\
0 & 1 & 2 & \ldots & n-1 \\
0 & 0 & 1 & \ldots & n-2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
n & n-1 & n-2 & \ldots & 1 \\
0 & n-1 & \ldots & 1 \\
0 & 0 & n & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & n
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & 2 & 0 & \ldots & 0 \\
1 & 2 & 3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & 3 & \ldots & n
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \alpha & 0 & \ldots & 0 \\
0 & 0 & \alpha & \ldots & 0 \\
0 & 0 & 0 & \alpha & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]
numbers:

1118. \[
\begin{pmatrix}
5 & 2 & -3 \\
4 & 5 & -4 \\
6 & 4 & -4
\end{pmatrix}
\]

1119. \[
\begin{pmatrix}
8 & 15 & -36 \\
8 & 21 & -48 \\
5 & 12 & -27
\end{pmatrix}
\]

1120. \[
\begin{pmatrix}
4 & 7 & -5 \\
-4 & 5 & 0 \\
1 & 9 & -4
\end{pmatrix}
\]

1121. \[
\begin{pmatrix}
4 & 2 & -5 \\
6 & 4 & -9 \\
5 & 3 & -7
\end{pmatrix}
\]

1122. Prove that if the last invariant factor \( E_n (\lambda) \) of the characteristic matrix \( A - \lambda E \) of a matrix \( A \) of order \( n \) is of the power \( n \), then all diagonal elements of distinct blocks of the Jordan form of the matrix \( A \) are distinct.

1123. Prove that a matrix \( A \) is nilpotent (i.e. \( A^k = 0 \) for any positive integer \( k \)) if and only if all its eigenvalues are zero.

1124. Prove that a nonzero nilpotent matrix cannot be brought to diagonal form by a similarity transformation.

1125. Find the Jordan form of the idempotent matrix \( A \) (a matrix is idempotent if it has the property \( A^2 = A \)).

1126. Prove that an involutory matrix \( A \) (that is, a matrix having the property \( A^2 = E \)) is similar to a diagonal matrix and find the form of this diagonal matrix.

1127. Prove that a periodic matrix \( A \) (that is, a matrix with the property that \( A^k = E \) for any positive integer \( k \)) is similar to a diagonal matrix and find the form of that diagonal matrix.

1128. Find the minimal polynomial (the definition of which is given in problem 830): (a) of a unit matrix, (b) of a zero matrix.

1129. For what matrices does the minimal polynomial have the form \( \lambda - \alpha \), where \( \alpha \) is a scalar?

1130. Find the minimal polynomial of a Jordan submatrix of order \( n \) with the number \( \alpha \) on the diagonal.

1131. Prove that the minimal polynomial of a block-diagonal matrix is equal to the lowest common multiple of the minimal polynomials of its diagonal blocks.

1132. Prove that the minimal polynomial of a matrix \( A \) is equal to the last invariant factor \( E_n (\lambda) \) of its characteristic matrix \( A - \lambda E \).

1133. Prove that some degree of the minimal polynomial of the matrix \( A \) is divisible by the characteristic polynomial of that matrix.

Find the minimal polynomials of the following matrices:

1134. \[
\begin{pmatrix}
3 & 1 & -1 \\
0 & 2 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]

1135. \[
\begin{pmatrix}
4 & -2 & 2 \\
-5 & 7 & -5 \\
-6 & 7 & -4
\end{pmatrix}
\]

1136. Prove that for two matrices to be similar it is necessary (but not sufficient) that they have the same characteristic and minimal polynomials. Give an example of two nonsimilar matrices which have the same characteristic polynomial \( \phi (\lambda) \) and the same minimal polynomial \( \psi (\lambda) \).

1137. Find the \( k \)th power \( A^k \) of the \( n \)th-order Jordan submatrix

\[
A = \begin{pmatrix}
\alpha & 1 & 0 & \ldots & 0 & 0 \\
0 & \alpha & 1 & \ldots & 0 & 0 \\
0 & 0 & \alpha & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \alpha & 1 \\
0 & 0 & 0 & \ldots & 0 & \alpha
\end{pmatrix}
\]

1138. Prove that the value of the polynomial \( f (x) \) in the Jordan submatrix \( A \) of order \( n \) with the number \( \alpha \) on the main diagonal

\[
A = \begin{pmatrix}
\alpha & 1 & 0 & \ldots & 0 \\
0 & \alpha & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & \alpha
\end{pmatrix}
\]

is given by the formula

\[
f (A) = \begin{pmatrix}
f (\alpha) & f' (\alpha) & f'' (\alpha) & f''' (\alpha) & \ldots & f^{(n-1)} (\alpha) \\
0 & f (\alpha) & f' (\alpha) & f'' (\alpha) & \ldots & f^{(n-2)} (\alpha) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & f (\alpha)
\end{pmatrix}
\]
1139. Solve problem 1080 by using the Jordan form of the matrix \( A \).
1140. Find the Jordan form of the square of a Jordan submatrix, the diagonal of which has the number \( \alpha \neq 0 \).
*1141. Find the Jordan form of the square of a Jordan submatrix with zero on the main diagonal (a nilpotent Jordan submatrix).
1142. Let \( X_i \) be the Jordan form of the matrix \( X \). Prove that \( (A + \alpha E)^{i} = A_i + \alpha E \), where \( A \) is any square matrix and \( \alpha \) is a scalar.
*1143. Find the Jordan form of the matrix
\[
\begin{pmatrix}
\alpha & 0 & 0 & \cdots & 0 \\
0 & \alpha & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \alpha
\end{pmatrix}
\]
of order \( n \geq 3 \).
*1144. Prove that any square matrix can be represented in the form of a product of two symmetric matrices, one of which is nonsingular.
*1145. Knowing the eigenvalues of the matrix \( A \), find the eigenvalues of the \( p \)th associated matrix \( A_p \) (the definition of which is given in problem 969).
1146. Knowing the eigenvalues of two square matrices, \( A \) of order \( p \) and \( B \) of order \( q \), find the eigenvalues of their Kronecker product \( A \times B \) (the definition of which is given in problem 963).
*1147. Let \( \psi ( \lambda ) = (\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_s)^{r_s} \) be the minimal polynomial of a matrix \( A \) of degree \( r = r_1 + r_2 + \cdots + r_s \). Here, \( r_h \) is the multiplicity of \( \lambda_h \) as a root of the minimal polynomial \( \psi ( \lambda ) \).
If the function \( f ( \lambda ) \) has the numbers
\[
\{ \lambda_1, \alpha_1 \}, \{ \lambda_2, \alpha_2 \}, \ldots, \{ \lambda_s, \alpha_s \},
\]
then we say that the function \( f ( \lambda ) \) is defined on the spectrum of the matrix \( A \), and the set of numbers (1) is called the set of values of the function \( f ( \lambda ) \) on the spectrum of the matrix \( A \). Prove that the values of the polynomials \( g ( \lambda ) = h ( \lambda ) \) at the matrix \( A \) coincide, that is, \( g (A) = h (A) \) if and only if the values of these polynomials coincide on the spectrum of \( A \).
*1148. Suppose a function \( f ( \lambda ) \) is defined on the spectrum of a matrix \( A \) (in the sense of the preceding problem). Prove that if there is at least one polynomial whose value on the spectrum of \( A \) coincides with the values of \( f ( \lambda ) \), then there is an infinity of such polynomials and there is only one of them that has a degree less than the degree of the minimal polynomial of matrix \( A \). This polynomial \( r ( \lambda ) \) is termed the Lagrange-Sylvester interpolation polynomial of the function \( f ( \lambda ) \) on the spectrum of the matrix \( A \). Its value of the matrix \( A \) is assumed, by definition, to be the value of the function \( f ( \lambda ) \) of that matrix: \( f (A) = r (A) \).
1149. Prove that if the function \( f ( \lambda ) \) is defined on the spectrum of a matrix \( A \) and the characteristic polynomial \( | A - \lambda E | \) does not have any multiple roots, then the Lagrange-Sylvester interpolation polynomial \( r ( \lambda ) \) exists and, hence, the matrix \( f (A) \) is meaningful. Find the aspects of \( r ( \lambda ) \) and \( f (A) \).
1150. Prove that if the function \( f ( \lambda ) \) is defined on the spectrum of a matrix \( A \) and the minimal polynomial of that matrix \( \psi ( \lambda ) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_s) \) has no multiple roots, then the Lagrange-Sylvester interpolation polynomial \( r ( \lambda ) \) exists and the matrix \( f (A) \) is meaningful. Find the expression for computing \( f (A) \).
*1151. Prove that if the function \( f ( \lambda ) \) is defined on the spectrum of a matrix \( A \), then the definition of the matrix \( f (A) \) (which was given in problem 1148) is meaningful, that is, there is a Lagrange-Sylvester interpolation polynomial \( r ( \lambda ) \). Let \( \psi ( \lambda ) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_s)^{r_s} \) be the minimal polynomial of the matrix \( A \), where the roots \( \lambda_1, \ldots, \lambda_s \) are distinct and
\[
\psi_k ( \lambda ) = \frac{\psi ( \lambda )}{(\lambda - \lambda_k)^{r_k}}, \quad (k = 1, 2, \ldots, s).
\]
Show that
\[
r ( \lambda ) = \sum_{k=1}^{s} [\alpha_{k, 1} + \alpha_{k, 2} (\lambda - \lambda_k) + \cdots + \alpha_{k, r_k} (\lambda - \lambda_k)^{r_k-1}] \psi_k ( \lambda ),
\]
where the numbers \( \alpha_{k, j} \) are defined by the equations
\[
\alpha_{k, j} = \frac{1}{(j-1)!} \left[ \left( \frac{f ( \lambda )}{\psi_k ( \lambda )} \right)^{(j-1)} \right]_{\lambda=\lambda_k}
\]
(\( j = 1, 2, \ldots, r_k; \ k = 1, 2, \ldots, s \)).
That is to say, the expression in square brackets in (1) is equal to the sum of the first \( r \) terms of the Taylor expansion in powers of the difference \( \lambda - \lambda_b \) for the function \( \psi_b(\lambda) \).

152. Let \( \psi(\lambda) = (\lambda - \lambda_1)^r(\lambda - \lambda_2)^s (\lambda - \lambda_3)^t \) be the minimal polynomial of a matrix \( A \) and let \( f(\lambda) \) be a function defined on the spectrum of that matrix. Write the expression for the matrix \( f(A) \) using the preceding problem.

153. Prove that if the matrices \( A \) and \( B \) are similar, and \( R = T^{-1}AT \) and also for the function \( f(\lambda) \) the matrix \( f(A) \) exists, then the matrix \( f(B) \) also exists and is similar to \( f(A) \), and we also have \( f(R) = T^{-1}f(A)T \) with the same matrix \( T \).

*154. Prove that if a matrix \( A \) is a block-diagonal matrix

\[
\begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}
\]

and the function \( f(\lambda) \) is defined on the spectrum of \( A \), then

\[
f(A) = \begin{pmatrix}
f(A_1) & 0 \\
0 & f(A_2)
\end{pmatrix}.
\]

155. Find the Lagrange-Sylvester interpolation polynomial and the value of the function \( f(\lambda) \) for the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

156. Show that if a matrix \( A \) is defined for all nonsingular matrices \( A \) and for them alone, then \( f(A) = A^{-1} \).

157. Show that if a matrix \( A \) is similar to the diagonal matrix

\[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]

and for the function \( f(\lambda) \) the matrix \( f(A) \) exists, then \( f(A) \) is also similar to the diagonal matrix, and we have

\[
f(A) = T^{-1} \begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix} T
\]

with the same matrix \( T \).

158. Prove that if \( f(\lambda) \) is defined on the spectrum of \( A \) and \( h(\lambda) \) exist, then the matrix \( f(A) \) also exists and we have \( f(A) = g(A) + h(A) \).

159. Prove that if \( f(\lambda) = g(\lambda)h(\lambda) \) and the matrices \( g(\lambda) \) and \( h(\lambda) \) exist, then the matrix \( f(\lambda) \) exists as well and we have \( f(\lambda) = g(\lambda)h(\lambda) \).

*160. Show that the function \( f(\lambda) = \frac{1}{\lambda} \) is defined for all nonsingular matrices \( A \) and for them alone, and that \( f(A) = A^{-1} \).

161. Show that if \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of a matrix \( A \) and the function \( f(\lambda) \) is meaningful for \( \lambda = \lambda_i \) then \( f(A) = f(\lambda_1)I + f(\lambda_2)D + \cdots + f(\lambda_n)D^n \) will be eigenvalues of the matrix \( f(A) \).

Compute the following values of the functions of the matrices, using the Lagrange-Sylvester interpolation polynomial and problems 1148 to 1152 or by finding the matrix in the Jordan form and applying problem 1160.

162. \( f(A) \), where \( A \) is given.

163. \( f(A) \), where \( A \) is given.

164. \( f(A) \), where \( A \) is given.
Sec. 15. Quadratic Forms

This section includes problems involving quadratic forms and also problems dealing with the properties of symmetric and orthogonal matrices that are connected with the theory of quadratic forms. We will use the following terminology:

A linear transformation is a transformation of the unknowns of the form

\[ x_1 = q_{11}y_1 + q_{12}y_2 + \ldots + q_{1n}y_n \\
\vdots \\
\vdots \\
\vdots \\
x_n = q_{n1}y_1 + q_{n2}y_2 + \ldots + q_{nn}y_n. \]

The matrix

\[ Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\
q_{21} & q_{22} & \cdots & q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}. \]

is made up of the coefficients of the transformation (1) in the appropriate order and is called the matrix of that transformation. A linear transformation is said to be nonsingular if its matrix is nonsingular. Two quadratic forms are said to be equivalent if one of them can be carried into the other via a nonsingular linear transformation. The canonical form of a given quadratic form is the equivalent form such that does not contain products of the unknowns; the normal form is a canonical form in which the coefficients of the squares of the unknowns (not counting such that are zero) are equal to \( \pm 1 \) for the real domain and \( \pm 1 \) for the complex domain.

Find the normal form of the following quadratic forms in the field of real numbers:

1175. \[ x_1^2 + 3x_2^2 + 2x_1x_2 + 2x_3x_4. \]
1176. \[ x_1^2 - 2x_2^2 + 2x_1x_2 + 2x_3x_4 + 2x_5x_6. \]
1177. \[ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2. \]
1178. \[ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2. \]
1179. \[ x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_3x_4 + 2x_5x_6 + 2x_2x_3 + 2x_4x_5 + 2x_6x_7. \]

Find the normal form and the nonsingular linear transformation that reduces to this form with respect to the following quadratic forms (the answer may differ from that given below; this is due to the ambiguous nature of the desired linear transformation):

1180. \[ x_1^2 + 5x_2^2 - 4x_3^2 - 2x_1x_2 - 4x_1x_3. \]
1181. \[ 4x_1^2 + x_2^2 - 4x_1x_2 + 4x_1x_3 - 3x_2x_3. \]
1182. \( x_1^2 + x_2^2 + 2x_3^2 + x_4^2 \)

1183. \( 2x_1^2 + 18x_2^2 + 8x_3^2 - 12x_1x_2 + 8x_1x_3 - 27x_2x_3 \)

1184. \(-12x_1^2 - 3x_2^2 - 12x_3^2 + 12x_1x_2 - 24x_1x_3 + 8x_2x_3 \)

1185. \( x_1^2 + x_2^2 + x_3^2 + 4x_1^2 \)

1186. \( 3x_1^2 + 2x_2^2 - x_3^2 - 2x_1^2 + 2x_1x_3 - 4x_2x_3 + 2x_3^2 \)

Reduce the following quadratic forms to canonical form with integral coefficients by means of a nonsingular linear transformation with rational coefficients and find the expression of the new unknowns in terms of the old unknowns:

1187. \( 2x_1^2 + 3x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 3x_2x_3 \)

1188. \( 2x_1^2 - 2x_2^2 + 2x_3^2 + 4x_1x_2 - 3x_1x_3 - x_2x_3 \)

1189. \( \frac{1}{2}x_1^2 + 2x_2^2 + 3x_3^2 - x_1x_2 + x_2x_3 - x_3^2 \)

Relative to the following quadratic forms, find the nonsingular linear transformation that carries the form \( f \) into the form \( g \) (the sought-for transformation is not defined uniquely):

1190. \( f = 2x_1^2 + 9x_2^2 + 3x_3^2 + 8x_1x_2 - 4x_1x_3 - 10x_2x_3 \)

\( g = 2y_1^2 + 3y_2^2 + 6y_3^2 - 4y_1y_2 - 4y_1y_3 + 8y_2y_3 \)

1191. \( f = 3x_1^2 + 10x_2^2 + 25x_3^2 - 12x_1x_2 - 18x_1x_3 + 40x_2x_3 \)

\( g = 5y_1^2 + 6y_2^2 + 12y_1y_2 \)

1192. \( f = 5x_1^2 + 5x_2^2 + 2x_3^2 + 8x_1x_2 + 6x_1x_3 + 6x_2x_3 \)

\( g = 4y_1^2 + y_2^2 + 9y_3^2 - 12y_1y_3 \)

Bring the following quadratic forms to canonical form and find the expression for the new unknowns in terms of the old ones (the answer is not unique):

1193. \( \sum_{j=1}^{n} a_j x_j^2 \), where not all the numbers \( a_1, a_2, \ldots, a_n \) are zero.

1194. \( \sum_{j=1}^{n} x_j^2 + \sum_{j < k} x_j x_k \).

1195. \( \sum_{j=1}^{n} x_j x_{j+1} \).

1196. \( \sum_{j=1}^{n} x_j x_{j+1} \).

1197. \( (x_1 - s)^2 \), where \( s = \frac{x_1 + x_2 + \ldots + x_n}{n} \).

1198. \( \sum_{j=1}^{n} [(j - i) x_j] \).

1199. Given the following quadratic form:

\[ f = l_1^2 + l_2^2 + \ldots + l_p^2 - l_{p+1}^2 - l_{p+2}^2 - \ldots - l_{p+q}^2 \]

where \( l_1, l_2, \ldots, l_p, l_{p+1}, l_{p+2}, \ldots, l_{p+q} \) are real linear forms in \( x_1, x_2, \ldots, x_n \). Prove that the positive index of inertia (that is, the number of positive coefficients in the canonical form) of the form \( f \) does not exceed \( p \), and the negative index of inertia does not exceed \( q \).

1200. Prove that if it is possible to pass from each of two forms \( f \) and \( g \) to the other by means of some (not necessarily nonsingular) linear transformation, then these forms are equivalent.

Determine which of the following forms are equivalent forms in the field of real numbers:

1201. \( f_1 = x_1^2 - x_1x_2, f_2 = y_1y_2 - y_2^2, f_3 = z_1^2 + z_2^2 \)

1202. \( f_1 = x_1^2 + 4x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 \)

\( f_2 = y_1^2 + 2y_2^2 - y_3^2 + 4y_1y_2 - 2y_1y_3 - 4y_2y_3 \)

\( f_3 = -4z_1^2 - z_2^2 - z_3^2 + 4z_1z_2 + 4z_2z_3 + 18z_3^2 \)

1203. Show that all quadratic forms in \( n \) unknowns may be split up into classes so that two forms will be equivalent if and only if they belong to one and the same class. Find the number of such classes in the complex and real domains.

1204. What values of rank and signature characterize those classes of real equivalent quadratic forms for which the form \( f \) is equivalent to the form \(-f\).

1205. Find the number of equivalence classes (in the domain of real numbers) of forms in \( n \) unknowns having a given signature \( s \).

1206. Prove that for a quadratic form to be decomposed into a product of two linear forms it is necessary and sufficient that the following conditions hold: (a) in the domain of real numbers: the rank does not exceed two, and for a rank of two the signature is zero; (b) in the domain of complex numbers: the rank does not exceed two.

1207. Prove that a quadratic form \( f \) is positive definite if and only if the matrix can be represented in the form \( A = C' C \), where \( C \) is a nonsingular real matrix and \( C' \) is the transpose of \( C \).

1208. Use problems 913, 951 and 1207 to prove that a quadratic form is positive definite if and only if all \( a \)
corner minors are positive. A corner minor of a quadratic form is a $k$th-order minor occupying the first $k$ rows and the first $k$ columns of the matrix ($k = 1, 2, \ldots, n$; $n$ is the order of the matrix).

1209. Prove that in a positive definite form all coefficients of the square of the unknowns are positive and that this condition is not sufficient for positive definiteness of the form.

*1210. Prove the following assertions:
(a) For a quadratic form $f$ to be positive definite, it is necessary and sufficient that all the principal minors (and not only the corner minors—see problem 1208) of the matrix be positive.
(b) For a quadratic form $f$ to be nonnegative (that is, $f \geq 0$ for all real values of the unknowns), it is necessary and sufficient that all principal minors of the matrix be nonnegative. Use examples to show that (unlike positive definite forms) it is not sufficient for the nonnegativeness of $f$ that all corner minors be nonnegative.
(c) For a real symmetric matrix $A$ to be represented in the form $A = C^TC$, where $C$ is a real nonsingular matrix, it is necessary and sufficient for all corner minors of the matrix $A$ to be positive.
(d) For a real symmetric matrix $A$ to be represented in the form $A = C^TC$, where $C$ is a real square matrix, it is necessary and sufficient that all principal minors of $A$ be nonnegative. Besides, if the rank of $A$ is equal to $r$, then the rank of $C$ is also equal to $r$, and we can assume the first $r$ rows of $C$ to be linearly independent and the remaining to be zero.

1211. Prove that a quadratic form $f$ is negative definite (i.e., $f < 0$ for all real values of the unknowns, not all of which are zero) if and only if the signs of the corner minors $D_1, D_2, \ldots, D_n$ alternate and $D_1 < 0$. Here, $D_k$ is the $k$th-order corner minor ($k = 1, 2, \ldots, n$).

Find all the values of the parameter $\lambda$ for which the following quadratic forms are positive definite:

1212. $2x_1^2 + x_2^2 + 2\lambda x_1 x_2 - 2x_1 x_2 - 2x_2 x_3$.
1213. $2x_1^2 + x_2^2 + 2x_1 x_2 + 2x_2 x_3$.
1214. $x_1^2 + 5x_2^2 + 2\lambda x_1 x_2 - 2x_1 x_2 + 4x_2 x_3$.
1215. $x_1^2 + 4x_2^2 + 2x_1 x_2 + 10x_1 x_2 + 6x_2 x_3$.

1216. $2x_1^2 + 2x_2^2 + x_3^2 + 2\lambda x_1 x_2 + 6x_1 x_2 + 2x_2 x_3$.

*1217. The discriminant $D_1$ of a quadratic form $f$ is the determinant of its matrix. Prove that if we add to a positive definite quadratic form $f \left(x_1, x_2, \ldots, x_n\right)$ the square of the nonzero linear form in the same unknowns, then the discriminant of the form will increase.

*1218. Let $f \left(x_1, x_2, \ldots, x_n\right) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be a positive definite quadratic form and let $q \left(x_2, x_3, \ldots, x_n\right) = \sum_{i=1}^{n} b_i x_i$. Prove that the inequality $D_1 \leq a_{11} D_0$ holds for the discriminants of these forms.

*1219. Prove that if a nonnegative quadratic form vanishes for even one nonzero set of real values of the unknowns, then that form is degenerate (that is, its discriminant is equal to zero).

*1220. We use the term composition of two quadratic forms

\[ f = \sum_{i,j=1}^{n} a_{ij} x_i x_j \quad \text{and} \quad g = \sum_{i,j=1}^{n} b_{ij} x_i x_j \]

for the quadratic form $(f, g) = \sum_{i,j=1}^{n} a_{ij} b_{ij} x_i x_j$.

Prove that
(a) if the forms $f$ and $g$ are nonnegative, then the form $(f, g)$ is nonnegative as well;
(b) if the forms $f$ and $g$ are positive definite, then the form $(f, g)$ is positive definite too.

1221. The term triangular transformation is used for a linear transformation of the form

\[ y_1 = x_1 + c_1 x_2 + \ldots + c_n x_n, \]
\[ y_2 = x_2 + c_2 x_3 + \ldots + c_n x_n, \]
\[ \vdots \]
\[ y_n = x_n. \]

Prove that
(a) a triangular transformation is nonsingular and the inverse of a triangular transformation is triangular;
(b) the corner minors $D_k \left(k = 1, 2, \ldots, n\right)$ (see problem 1208) of a quadratic form $f = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ remain unchanged under a triangular transformation.
1222. Prove that
(a) for the quadratic form of rank \( r \), \( f = \sum_{i,j=1}^{n} a_{ij}x_{i}x_{j} \) to be brought to the form

\[ f = \lambda_{1}y_{1}^{2} + \ldots + \lambda_{r}y_{r}^{2}, \]

where \( \lambda_{k} \neq 0 \) (\( k = 1, 2, \ldots, r \)), by a triangular transformation, it is necessary and sufficient that

\[ D_{k} \neq 0 \quad (k = 1, 2, \ldots, r), \quad D_{k} = 0 \quad (k > r), \]

where \( D_{k} \) (\( k = 1, 2, \ldots, n \)) are corner minors of the form \( f \) (see problem 1208);

(b) the indicated canonical form (1) is defined uniquely and its coefficients can be found from the formulas

\[ \lambda_{k} = \frac{D_{k}}{D_{k-1}} \quad (k = 1, 2, \ldots, r; D_{0} = 1) \]

(Sylvester's theorem).

1223. Let the corner minors of the quadratic form \( f \) of rank \( r \) satisfy the conditions (2) of the preceding problem. Prove that the positive index of inertia of this form is equal to the number of preservations of sign, and the negative index is equal to the number of changes of sign in the following set of numbers:

\[ 1 = D_{0}, D_{1}, D_{2}, \ldots, D_{r}. \]

Determine that there is one positive definite form in the following pairs of quadratic forms; find the nonsingular linear transformation that brings that form to the normal form and that brings the other form to canonical form; then write the canonical form (the linear transformation is not defined uniquely):

1224. \( f = -5x_{1}x_{2}, \quad g = x_{1}^{2} - 2x_{1}x_{2} + 4x_{2}^{2}, \)
1225. \( f = x_{1}^{2} + 26x_{2}^{2} + 10x_{1}x_{2}, \quad g = x_{1}^{2} + 56x_{2}^{2} + 16x_{1}x_{2}, \)
1226. \( f = -8x_{1}^{2} - 28x_{2}^{2} + 14x_{1}^{2} + 16x_{1}x_{2} + 14x_{1}x_{2} + 32x_{2}x_{3}, \quad g = x_{1}^{2} + 4x_{2}^{2} + 2x_{1}^{2} + 2x_{1}x_{2}. \)

1227. \( f = 2x_{1}^{2} + x_{1}x_{2} + x_{1}x_{3} - 2x_{2}x_{3} + 2x_{3}x_{4}, \quad g = \frac{1}{4} x_{1}^{2} + x_{2}^{2} + x_{3}^{2} + 2x_{4}^{2} - 2x_{2}x_{3}, \)
1228. \( f = x_{1}^{2} + x_{2}^{2} - 2x_{1}x_{2} + 4x_{1}x_{2}, \quad g = x_{1}^{2} + \frac{8}{3} x_{2}^{2} - x_{3}^{2} + 2x_{4}^{2} - 2x_{2}x_{3}, \)
1229. \( f = x_{1}^{2} - 15x_{2}^{2} + 4x_{1}x_{4} - 2x_{1}x_{3} - 6x_{2}x_{3}, \quad g = x_{1}^{2} + 17x_{2}^{2} + 3x_{3}^{2} + 4x_{1}x_{2} - 2x_{1}x_{3} - 14x_{3}x_{4}. \)

1230. Suppose we have a pair of forms, \( f \) and \( g \), in the same unknowns, and \( g > 0 \). Prove that the canonical form

\[ f = \lambda_{1}y_{1}^{2} + \ldots + \lambda_{r}y_{r}^{2} \]

obtained for \( f \) under any linear transformation that brings \( g \) to the normal form (that is, to a sum of squares) is determined uniquely up to the order of the terms, and the coefficients \( \lambda_{1}, \lambda_{2}, \ldots, \lambda_{r} \) are roots of the so-called \( \lambda \)-equation of the pair of forms \( f, g \), namely, of the equation \( |A - \lambda B| = 0 \), where \( A \) and \( B \) are the respective matrices of the forms \( f \) and \( g \).

Is it possible to bring the following pairs of quadratic forms to canonical form by means of a single real nonsingular linear transformation?

1231. \( f = x_{1}^{2} + 4x_{1}x_{2} - x_{2}^{2}, \quad g = x_{1}^{2} + 6x_{1}x_{2} + 5x_{2}^{2}, \)
1232. \( f = x_{1}^{2} + x_{1}x_{2} - x_{2}^{2}, \quad g = x_{1}^{2} - 2x_{1}x_{2}, \)

1233. Suppose we have two positive definite forms \( f \) and \( g \) and suppose one nonsingular linear transformation of the unknowns brings the form \( f \) to \( \sum_{i=1}^{n} \lambda_{i}y_{i}^{2} \) and the form \( g \) to the normal form, and the second transformation does the opposite, bringing the form \( f \) to the normal form and the form \( g \) to \( \sum_{i=1}^{n} \mu_{i}y_{i}^{2} \). Find the relationship between the coefficients \( \lambda_{1}, \ldots, \lambda_{n} \) and \( \mu_{1}, \ldots, \mu_{n} \).
Without seeking the linear transformation, find the canonical form of the given form \( f \) to which it will be carried by the transformation that brings the other given form \( g > 0 \) to the normal form:

1234. \( f = 21x_1^2 - 18x_2^2 + 6x_3^2 + 4x_1x_2 + 28x_1x_3 + 6x_2x_3, \)
\[ g = 11x_1^2 - 6x_2^2 + 6x_3^2 - 12x_1x_2 + 12x_1x_3 - 6x_2x_3, \]

1235. \( f = 14x_1^2 - 4x_2^2 + 17x_3^2 + 8x_1x_2 - 40x_1x_3 - 26x_2x_3, \)
\[ g = 9x_1^2 + 6x_2^2 + 6x_3^2 + 12x_1x_2 - 10x_1x_3 - 2x_2x_3. \]

1236. Prove that the two pairs of forms \( f_1, g_1 \) and \( f_2, g_2 \), where \( g_1 \) and \( g_2 \) are positive definite, are equivalent (i.e., there exists a nonsingular linear transformation that carries \( f_1 \) into \( f_2 \) and \( g_1 \) into \( g_2 \)) if and only if the roots of their characteristic equations \( |A_1 - \lambda B_1| = 0 \) and \( |A_2 - \lambda B_2| = 0 \) coincide.

Determine whether the following pairs of forms are equivalent without finding the linear transformation of one pair into the other:

1237. \( f_1 = 2x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_2 + 2x_1x_3, \)
\[ g_1 = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_3, \]
\[ f_2 = 2x_1^2 + 5x_2^2 + 2x_3^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3, \]
\[ g_2 = x_1^2 + x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3. \]

1238. \( f_1 = 4x_1^2 + 6x_2^2 + 21x_3^2 + 4x_1x_2 - 4x_1x_3 - 22x_2x_3, \)
\[ g_1 = 4x_1^2 + 3x_2^2 + 6x_3^2 + 4x_1x_2 - 4x_1x_3 - 6x_2x_3, \]
\[ f_2 = 9x_1^2 + 6x_2^2 + 6x_3^2 - 6x_1x_2 - 6x_1x_3 + 12x_2x_3, \]
\[ g_2 = 9x_1^2 + 3x_2^2 + 3x_3^2 - 6x_1x_2 - 6x_1x_3 + 2x_2x_3. \]

Find the nonsingular linear transformation that carries the pair of quadratic forms \( f_1, g_1 \) into the other pair \( f_2, g_2 \) (the desired transformation is not defined uniquely):

1239. \( f_1 = 2x_1^2 - 7x_2^2 + 2x_1x_2, f_2 = -7y_1^2 - 3y_2^2 - 12y_1y_2, \)
\[ x_1 = -2y_1^2 + 13y_2^2 + 10y_1y_2, x_2 = 13y_1^2 + 25y_2^2 + 36y_1y_2, \]

1240. \( f_1 = 3x_1^2 + 2x_2^2 - 10x_1x_2, f_2 = -9y_1^2 - 20y_2^2 - 44y_1y_2, \)
\[ x_1 = 2y_1^2 + 3x_2^2 - 6x_1x_2, x_2 = 29y_1^2 + 4y_2^2 + 20y_1y_2. \]

1241. Suppose \( f(x_1, x_2, \ldots, x_n) \) and \( g(x_1, x_2, \ldots, x_n) \) are two quadratic forms, at least one of which is positive definite. Prove that the "surfaces" \( f = 1 \) and \( g = 1 \) in \( n \)-dimensional space do not intersect (that is, do not have any points in common) if and only if the form \( f - g \) is definite.

*1242. Prove that the canonical form \( \sum_{i=1}^{n} \lambda_i x_i^2 \) to which the quadratic form \( f \) is reduced by an orthogonal transformation is uniquely defined and its coefficients \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are roots of the characteristic equation \( |A - \lambda E| = 0 \) of the matrix \( A \) of the form \( f \).

Find the canonical form to which the following quadratic forms can be reduced by means of an orthogonal transformation (do not seek the transformation itself):

1243. \( 3x_1^2 + 3x_2^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3, \)
1244. \( 7x_1^2 + 7x_2^2 + 7x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3, \)
1245. \( x_1^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3, \)
1246. \( 3x_1^2 + 3x_2^2 - 6x_1x_2 + 4x_2x_3, \)
1247. \( \sum_{k=1}^{n-1} x_kx_{k+1}. \)

Find the orthogonal transformation that brings the following forms to canonical form (reduction to principal axes) and write that canonical form (the transformation is not defined uniquely):

1248. \( 6x_1^2 + 5x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_1x_3, \)
1249. \( 11x_1^2 + 5x_2^2 + 2x_3^2 + 16x_1x_2 + 4x_1x_3 - 20x_2x_3, \)
1250. \( x_1^2 + x_2^2 + 5x_3^2 - 6x_1x_2 - 2x_1x_3 + 2x_2x_3, \)
1251. \( x_1^2 + x_2^2 + 2x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3, \)
1252. \( 14x_1^2 + 14x_2^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3, \)
1253. \( x_1^2 - 5x_2^2 + x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3, \)
1254. \( 8x_1^2 - 7x_2^2 + 8x_3^2 + 8x_1x_2 - 2x_1x_3 + 8x_2x_3, \)
1255. \( 2x_1x_2 - 6x_1x_3 - 6x_2x_4 + 2x_3x_4. \)
Find the canonical form to which the following forms can be brought by an orthogonal transformation and express the new unknowns in terms of the old unknowns (the sought transformation is not defined uniquely):

1263. \[ \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j. \]

1264. \[ \sum_{i=1}^{n} x_i x_j. \]

*1265. We will say that two quadratic forms are orthogonally equivalent if we can pass from one to the other via an orthogonal transformation. Prove that for two forms to be orthogonally equivalent it is necessary and sufficient that the characteristic polynomials of their matrices coincide.

Determine which of the following quadratic forms are orthogonally equivalent:

1266. \[ f = 9x_1^2 + 9x_3^2 + 12x_1 x_2 + 12x_1 x_3 - 6x_2 x_3; \]
\[ g = -3y_1^2 + 6y_2^2 - 3y_3^2 - 12y_1 y_2 + 12y_1 y_3 + 6y_2 y_3; \]
\[ h = 11z_1^2 - 4z_2^2 + 11z_2^2 + 8z_1 z_2 + 8z_1 z_3 + 8z_2 z_3. \]

1267. \[ f = 7x_1^2 + x_2^2 + x_3^2 - 8x_1 x_2 - 8x_1 x_3 + 16x_2 x_3; \]
\[ g = \frac{2}{3} y_1^2 - \frac{1}{3} y_2^2 - \frac{1}{3} y_3^2 - \frac{2}{3} y_1 y_2 + \frac{4}{3} y_1 y_3 + \frac{8}{3} y_2 y_3; \]
\[ h = \frac{2}{3} z_1^2 - \frac{1}{3} z_2^2 + 2 \sqrt{2} z_1 z_2. \]

1268. Prove that any real symmetric matrix \( A \) may be represented as \( A = Q^{-1} B Q \), where \( Q \) is an orthogonal matrix and \( B \) is a real diagonal matrix.

For the following matrices, find the orthogonal matrix \( Q \) and the diagonal matrix \( B \) such that the given matrix represented as \( Q^{-1} B Q \):

1269. \[
\begin{pmatrix}
3 & 2 & 0 \\
2 & 4 & 0 \\
0 & -2 & 5
\end{pmatrix}
\]
1270. \[
\begin{pmatrix}
2 & 2 & 1 \\
2 & 3 & 4 \\
0 & -1 & 5
\end{pmatrix}
\]

*1271. Prove that all the eigenvalues of a real symmetric matrix \( A \) lie on the interval \([a, b]\) if and only if the quadratic form with matrix \( A - \lambda E \) is positive definite for \( \lambda < a \) and is negative definite for any \( \lambda > b \).

*1272. Let \( A \) and \( B \) be real symmetric matrices. Prove that if the eigenvalues of \( A \) lie on the interval \([a, b]\), and the eigenvalues of \( B \) lie on the interval \([c, d]\), then the eigenvalues of the matrix \( A + B \) lie on the interval \([a + c, b + d]\).

1273. Prove that a nonsingular quadratic form can be brought to normal form by an orthogonal transformation if and only if the matrix is orthogonal.

1274. Prove that the matrix of a positive definite quadratic form is orthogonal if and only if that form is a sum of squares. Formulate this statement in the language of matrices.

*1275. Prove that any real nonsingular matrix \( A \) can be represented in the form \( A = Q B \), where \( Q \) is an orthogonal matrix and \( B \) is a triangular matrix that looks like

\[
\begin{pmatrix}
b_{11} & b_{12} & b_{13} & \ldots & b_{1n} \\
0 & b_{22} & b_{23} & \ldots & b_{2n} \\
0 & 0 & b_{33} & \ldots & b_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & b_{nn}
\end{pmatrix}
\]

with positive elements on the main diagonal; prove that this representation is unique.

*1276. Prove that

(a) any real nonsingular matrix \( A \) may be represented as \( A = Q_1 B_1 \) and also as \( A = Q_2 B_2 \), where the matrices \( Q_1 \) and \( Q_2 \) are real and orthogonal and the matrices \( B_1 \) and \( B_2 \) are real, symmetric and with positive corner minors. Each of these representations is unique.
(1) any complex nonsingular matrix \( A \) may be represented both as \( A = QB \) and as \( A = BQ \), where the matrices \( Q \) and \( Q \) are unitary and the matrices \( B \) and \( B \) are Hermitian and with positive corner minors (the matrix \( B \) is said to be Hermitian if \( B^* = B \)). Each of these representations is unique;

(2) let \( A \) be a symmetric (or Hermitian) matrix with positive corner minors and let \( B \) be an orthogonal (respectively, unitary) matrix. Prove that

(1) the products \( AB \) and \( BA \) are symmetric (Hermitian) matrices with positive corner minors if and only if \( B \) is an orthogonal matrix.

(2) the products \( AB \) and \( BA \) are orthogonal (unitary) if and only if \( A \) is a unit matrix.

Chapter IV

Vector Spaces and Their Linear Transformations

Sec. 16. Affine Vector Spaces

The following notation will be used here: vectors are denoted by lower-case boldface Latin letters, vector spaces and their subspaces and linear manifolds are denoted by upper-case boldface Latin letters. In ordinary notation, the coordinates of a vector are written out linearly in parentheses; for example, \( x = (x_1, x_2, \ldots, x_n) \). In matrix notation, the vectors of the basis are written linearly in parentheses, while the coordinates of the vectors are written in a column in parentheses.

The transition matrix from an old basis \( e_1, e_2, \ldots, e_n \) to a new basis \( e'_1, e'_2, \ldots, e'_n \) is the matrix \( T = (t_{ij}) \), the columns of which contain the coordinates of the new basis vectors in terms of the old basis. Thus, the old and new bases are connected by the matrix equation

\[
(e'_1, e'_2, \ldots, e'_n) = (e_1, e_2, \ldots, e_n) \cdot T.
\]

In this notation, the coordinates \( x_1, x_2, \ldots, x_n \) of the vector \( x \) in the old basis are connected with the coordinates \( x'_1, x'_2, \ldots, x'_n \) of the same vector in the new basis by the equations \( x'_i = \sum_{j=1}^{n} t_{ij} x_j \) or, in matrix notation, by

\[
\begin{pmatrix}
x'_1 \\
x'_2 \\
\vdots \\
x'_n
\end{pmatrix} = T
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}.
\]

A linear subspace of a vector space \( V \) is a nonempty subset \( F \) of \( V \) which means it contains at least one vector of \( V \) and in \( F \) that have the following properties:

(1) the sum \( x + y \) of any two vectors in \( F \) is also in \( F \);
(2) the product \( ax \) of any vector \( x \) in \( L \) by any scalar \( a \) lies in \( L \).

A linear manifold \( P \) of a vector space \( R \) is a collection \( P \) of vectors in \( R \) obtained by adjoining the same vector \( x_0 \) to all vectors of some subspace \( L \) of \( R \). This relationship between \( L \) and \( P \) will be denoted thus: \( P = L \oplus x_0 \) or \( L = P - x_0 \). We will say that the linear manifold \( P \) has been obtained from the linear subspace \( L \) by a parallel shift by the amount of the vector \( x_0 \).

The dimension of a linear manifold is the dimension of that linear subspace whose parallel translation yields the given manifold. The correctness of this definition follows from the assertion of problem 1331. One dimensional linear manifolds will be called straight lines, two dimensional ones will be termed planes.

The sum of two linear subspaces \( L_1 \) and \( L_2 \) of a vector space \( R \) is the collection \( S = L_1 \oplus L_2 \) of all vectors of \( R \), each of which is represented in the form \( x = x_1 + x_2 \), where \( x_1 \in L_1 \) and \( x_2 \in L_2 \). Here, the notation \( a \in A \) means that the element \( a \) is a member of the set \( A \). The intersection of two linear subspaces \( L_1 \) and \( L_2 \) of a vector space \( R \) is the collection \( D = L_1 \cap L_2 \) of all vectors of \( R \), each of which belongs both in \( L_1 \) and in \( L_2 \).

The direct sum of two linear subspaces \( L_1 \) and \( L_2 \) of a vector space \( R \) is the sum of those subspaces, provided their intersection consists solely of the zero vector, that is, \( L_1 \cap L_2 = 0 \). In the case of a direct sum we will write \( S = L_1 \oplus L_2 \).

An n-dimensional vector space will be denoted as \( R_n \).
And, unless otherwise stated, it will be assumed that the basis field is the field of real numbers, that is, \( R_n \) consists of all \( n \)-dimensional vectors with arbitrary real coordinates.

The vectors \( e_1, e_2, \ldots, e_n \) and \( x \) are specified by their components (coordinates) in terms of some basis. Show that the vector \( e_1, e_2, \ldots, e_n \), themselves form a basis, and then find the coordinates of the vector \( x \) in that basis:

1277. \( e_1 = (1, 1, 1), e_2 = (1, 2, 3), e_3 = (1, 2, 3); x = (6, 9, 14) \)

1278. \( e_1 = (2, 3, 3), e_2 = (3, 2, 3), e_3 = (1, 4, 1); x = (6, 2, 7) \)

1279. \( e_1 = (1, 2, 1, 2), e_2 = (1, 2, 1, 4), e_3 = (1, 3, 1, 0); x = (5, 13, 4) \)

Prove that each of the two sets of vectors is a basis and find the relationship between the coordinates of a vector in one basis and the vector in the two bases:

1280. \( e_1 = (1, 2, 1), e_2 = (2, 3, 3), e_3 = (3, 7, 1) \)

1281. \( e_1 = (1, 1, 1, 1), e_2 = (2, 1, 1, 1), e_3 = (1, 1, 1) \)

1282. Find the coordinates of the polynomial \( f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \)
(a) in the basis \( 1, x, x^2, \ldots, x^n \)
(b) in the basis \( 1, x, (x - 1)^2, \ldots, (x - 1)^n \) having ascertained that the last two polynomials do indeed form a basis.

1283. Find the transition matrix from the basis \( 1, x, x^2, x^3, \ldots, x^n \) to the basis \( 1, x, (x - 1)^2, \ldots, (x - 1)^n \) in the space of polynomials of degree less than or equal to \( n \).

1284. How does the transition matrix vary from one basis to another?
(a) if two vectors of the first basis are interchanged?
(b) if two vectors of the second basis are interchanged?
(c) if the vectors in both bases are given in reverse order?

State whether each of the following sets of vectors is a linear subspace of the appropriate vector space (from 1285 to 1293):

1285. All vectors of an n-dimensional vector space whose coordinates are integers.

1286. All vectors of a plane, each of which lies on one of the coordinate axes Ox or Oy.

1287. All vectors of a plane, the endpoints of the vectors lying on a given straight line (unless otherwise stated, the origin of any vector is assumed to coincide with the coordinate origin).

1288. All vectors of a plane, the origin and terminal point of each vector lying on a given straight line.
1289. All vectors of three-dimensional space, the termini of the vectors not lying on a given straight line.
1290. All vectors of a plane, the termini of the vectors lying in the first quadrant of the coordinate system.
1291. All vectors in $\mathbb{R}^n$, whose coordinates satisfy the equation $x_1 + x_2 + \ldots + x_n = 0$.
1292. All vectors in $\mathbb{R}^n$, whose coordinates satisfy the equation $x_1 + x_2 + \ldots + x_n = 1$.
1293. All vectors that are linear combinations of the given vectors $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^n$.
1294. Enumerate all linear subspaces of three-dimensional vector space.
1295. Suppose the linear subspace $L_1$ lies in the linear subspace $L_2$. Prove that the dimension of $L_1$ does not exceed the dimension of $L_2$, the dimensions being equal if and only if $L_1 = L_2$. Does this assertion hold true for any two linear subspaces of the given space?
1296. Prove that if the sum of the dimensions of two linear subspaces of $n$-dimensional space exceeds $n$, then these subspaces have a nonzero vector in common.
1297. Prove that the following sets of vectors form linear subspaces and find the basis and dimension of each (1297 to 1300):
1298. All $n$-dimensional vectors with the first and last coordinates equal.
1299. All $n$-dimensional vectors whose coordinates with non-number labels are zero.
1300. All $n$-dimensional vectors whose coordinates with non-number labels are equal.
1301. All $n$-dimensional vectors of the form $(\alpha, \beta, \alpha, \beta, \ldots, \alpha, \beta)$, where $\alpha$ and $\beta$ are any numbers.
1302. Prove that all square matrices of order $n$ with real number elements taken from any field $F$ form a vector space over the field of real numbers (respectively, over any field $F$ if the operations involved are addition of matrices and multiplication of a matrix by a scalar). Find the dimension of that space.
1303. Prove that all symmetric matrices form a linear subspace of the space of all $n \times n$ matrices. Find the basis and dimension of that subspace.
1304. Prove that skew-symmetric matrices form a linear subspace of all $n \times n$ matrices. Find the basis and dimension of that subspace.
1305. Prove that if a linear subspace $L$ of the space of polynomials of degree $\leq n$ contains at least one polynomial of degree $k$ for $k = 0, 1, 2, \ldots, p$, but does not contain polynomials of degree $k > p$, then it coincides with the subspace $L_p$ of all polynomials of degree $\leq p$.
1306. Let $f$ be a nonnegative quadratic form in $n$ unknowns of rank $r$. Prove that all solutions of the equation $f = 0$ form an $(n - r)$-dimensional linear subspace of the space $\mathbb{R}^n$.
1307. Prove that the solutions of any system of homogeneous linear equations in $n$ unknowns of rank $r$ form a linear subspace of the $n$-dimensional space $\mathbb{R}^n$ of dimension $d = n - r$ and, conversely, for any linear subspace $L$ of dimension $d$ of the space $\mathbb{R}^n$, there exists a system of homogeneous linear equations in $n$ unknowns of rank $r = n - d$, whose solutions fill the given subspace $L$ exactly.
1308. Find some basis and the dimension of a linear subspace $L$ of the space $\mathbb{R}^n$ if $L$ is specified by the equation $x_1 + x_2 + \ldots + x_n = 0$.
1309. Prove that the dimension of a linear subspace $L$ spanned by the vectors $x_1, x_2, \ldots, x_k$ (that is, the subspace of all linear combinations of the given vectors) is equal to the rank of the matrix made up of the coordinates of the given vectors in some basis; for the basis of the subspace we can take any maximal linearly independent subset of the set of given vectors.
1310. Find the dimensions and bases of the linear subspaces spanned by the following sets of vectors:
1311. $a_1 = (1, 0, 0, 0, \ldots, 0), a_2 = (2, 1, 1, 0), a_3 = (1, 1, 1, 0), a_4 = (1, 2, 3, 4), a_5 = (0, 1, 2, 3)$.
1312. \( a_1 = (1, -1, 1, 0), \ a_2 = (1, 1, 0, 1), \ a_3 = (2, 0, 1, 1) \).

1313. \( a_1 = (1, -1, 1, -1), \ a_2 = (1, 1, 0, 3), \ a_3 = (3, 1, 1, -1, 7), \ a_4 = (0, 2, -1, 1, 2) \).

1314. Prove that the union of the intersection of two linear subspaces of the space \( R_n \) are themselves linear subspaces of that space.

1315. Prove that the union (or sum) \( S = L_1 + L_2 \) of two linear subspaces of the space \( R_n \) is equal to the intersection of all linear subspaces in \( R_n \) that contain both \( L_1 \) and \( L_2 \).

1316. Prove that the union of two linear subspaces of \( R_n \) is equal to the dimension of the union plus the dimension of the intersection of these subspaces.

Find the dimension \( s \) of the union and the dimension \( d \) of the intersection of the linear subspaces: \( L_1 \) spanned by the vectors \( a_1, a_2, \ldots, a_k \) and \( L_2 \) spanned by the vectors \( b_1, b_2, \ldots, b_l \):

1317. \( a_1 = (1, 2, 0, 1), \ a_2 = (1, 1, 1, 0) \) and \( b_1 = (1, 0, 1, 0), \ b_2 = (1, 3, 0, 1) \).

1318. \( a_1 = (1, 1, 1, 1), \ a_2 = (1, -1, 1, -1), \ a_3 = (3, 1, 1, 3), \ b_1 = (1, 2, 0, 2), \ b_2 = (1, 2, 1, 2), \ b_3 = (3, 1, 3, 1) \).

1319. Suppose \( L_1 \) is a linear subspace of the space \( R_n \) with basis \( a_1, a_2, \ldots, a_k \) and let \( L_2 \) be a linear subspace of the same space with basis \( b_1, b_2, \ldots, b_l \).

Prove the following rules for constructing the basis of the union \( S = L_1 + L_2 \) and the basis of the intersection \( D \) of these subspaces:

1. The basis of the union \( S \) is the maximum linearly independent subset of the set of vectors \( a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l \). Constructing it reduces to computing the rank of the matrix made up of the coordinates of this last set of vectors.

2. The basis of the union \( S \) may be obtained by adjoining to the linearly independent vectors \( a_1, a_2, \ldots, a_k \) certain of the vectors \( b_1, \ldots, b_l \) (see problem 659). By changing, if necessary, the order of these vectors, we can consider that the vectors \( a_1, \ldots, a_k, b_1, \ldots, b_{k+l} \) form a basis of \( S \).

The equation

\[ x_1 a_1 + \ldots + x_k a_k = y_1 b_1 + \ldots + y_l b_l \]

is equivalent to a system of \( n \) homogeneous linear equations in \( k + l \) unknowns \( x_1, \ldots, x_k, y_1, \ldots, y_l \) of rank \( d \). Since the first \( s \) columns of the matrix of the system are linearly independent and, hence, at least one minor order \( s \) in these columns is different from zero, we can take the last \( k + l - s = d \) unknowns \( y_{s+k+1}, \ldots, y_l \) for the free unknowns. We can therefore find, for the system of equations (1), the following fundamental set of solutions

\[ x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_l \ (i = 1, 2, \ldots, d) \]

such that

\[ \begin{vmatrix} y_{1, s-k+1} & \ldots & y_{1, l} \\ \vdots & \ddots & \vdots \\ y_{d, s-k+1} & \ldots & y_{d, l} \end{vmatrix} \neq 0. \]

Then the basis of the intersection \( D \) is the set of vectors

\[ c_i = \sum_{j=1}^{l} y_{ij} b_j \ (i = 1, 2, \ldots, d). \]
1324. Let $L$, $L_1$, $L_2$ be linear subspaces of the space $R_n$. Prove that $L$ is a direct sum of $L_1$, $L_2$ if and only if the following conditions hold true:

(a) $L$ contains $L_1$ and $L_2$;
(b) every vector $x \in L$ is uniquely represented as $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$. In other words, the sum $L = L_1 + L_2$ is a direct sum if and only if for every vector $x \in L$, the representation $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$, is unique.

1325. Prove that the sum $S$ of linear subspaces $L_1$ and $L_2$ is a direct sum if and only if at least one vector $z \in S$ can be uniquely represented as $z = z_1 + z_2$, where $z_1 \in L_1$, $z_2 \in L_2$.

1326. Suppose a linear subspace $L$ is a direct sum of the linear subspaces $L_1$ and $L_2$. Prove that the dimension of $L$ is equal to the sum of the dimensions of $L_1$ and $L_2$; also, any bases of $L_1$ and $L_2$ together yield a basis of $L$.

1327. Prove that for any linear subspace $L_1$ of the space $R_n$, it is possible to find another subspace $L_2$ such that the whole space $R_n$ will be a direct sum of $L_1$ and $L_2$.

1328. Prove that the space $R_n$ is a direct sum of two linear subspaces: $L_1$ specified by the equation $x_1 + x_2 + \ldots + x_n = 0$ and $L_2$ given by the system of equations $x_1 = \ldots = x_n$. Find the projections of the unit vectors $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, 0, 0, \ldots, 1)$ on $L_2$ parallel to $L_1$ and on $L_1$ parallel to $L_2$.

1329. Prove that the space of all square matrices of order $n$ is a direct sum of the linear subspaces $L_1$ of symmetric matrices and $L_2$ of skew-symmetric matrices. Find the projections $A_1$, and $A_2$ of the matrix

$$
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

on $L_1$ parallel to $L_2$, on $L_1$ parallel to $L_2$, and on $L_2$ parallel to $L_1$.

1330. Prove that the solution of any consistent system of linear equations over a linear algebra is a linear manifold in the space $R_n$ of dimension $d = n - r$.

1331. Given two linear manifolds (see Sec. 10, introduction) $P_1 = L_1 + \mathbf{a}$ and $P_2 = L_2 + \mathbf{b}$, where $L_1$, $L_2$ are linear subspaces and $\mathbf{a}$, $\mathbf{b}$ are vectors of the space $R_n$. Prove that $P_1 = P_2$ if and only if $L_1 = L_2$ and $\mathbf{a} - \mathbf{b} \in L_1 \cap L_2$. Thus, the linear space whose parallel translation yields the given manifold is defined uniquely.

1332. Prove that if $P = L + x_0$, where $L$ is a linear subspace and $x_0$ is a vector of the space $R_n$, then the vector $x_0$ belongs to the manifold $P$ and replacing this vector by any other vector $x \in P$ yields the same manifold $P$.

1333. Prove that if a straight line has two common points with a linear manifold, then it lies entirely in that manifold (here, a point is identified with a vector having the same coordinates, i.e. issuing from the coordinate origin to the given point).

1334. Prove that any two straight lines of the space $R_n$ ($n > 3$) are contained in a certain three-dimensional linear manifold lying in $R_n$.

1335. Find the conditions that are necessary and sufficient for the two straight lines $x = a_0 + a_1t$ and $x = b_0 + b_1t$ of the space $R_n$ ($n \geq 1$) to lie in a single plane.

1336. Find the necessary and sufficient conditions for two straight lines $x = a_0 + a_1t$ and $x = b_0 + b_1t$ to pass through a single point without being coincident. Indicate a method for finding the point of intersection of these straight lines.

Find the point of intersection of the two straight lines $a_0 + a_1t$ and $b_0 + b_1t$:

1337. $a_0 = (2, 1, 1, 3, -3)$, $a_1 = (2, 3, 1, 1, -1)$; $b_0 = (1, 2, 1, 1, 2)$, $b_1 = (1, 2, 1, 0, 1)$.

1338. $a_0 = (3, 1, 2, 1, 3)$, $a_1 = (1, 0, 1, 1, 2)$; $b_0 = (2, 2, -1, -1, -2)$, $b_1 = (2, 1, 0, 1, 1)$.

1339. Find the necessary and sufficient conditions for drawing through a point specified by a vector $r$ a unique straight line intersecting two given straight lines $x = a_0 + a_1t$ and $x = b_0 + b_1t$. Indicate a method for constructing...
such a straight line and the points of its intersection with the given straight lines.

Find a straight line that passes through a point specified by a vector \( c \) and intersects the straight lines \( x = a_0 + a_1 t \), \( x = b_0 + b_1 t \), and find the points of intersection of the desired straight line with the two given straight lines.

1340. \( a_0 = (1, 0, -2, 1) \), \( a_1 = (1, 2, -4, -5) \); \( b_0 = (0, 1, 1, -1) \); \( b_1 = (2, 3, -2, -4) \); \( c = (8, 9, -11, -15) \).

1341. \( a_0 = (1, 1, 1, 1) \), \( a_1 = (1, 2, 1, 0) \); \( b_0 = (2, 2, 3, 1) \); \( b_1 = (1, 0, 1, 3) \); \( c = (4, 5, 2, 7) \).

1342. Prove that any two planes of the space \( R_n \) are contained in a linear manifold of dimension \( \leq 5 \).

1343. Prove that two linear manifolds of the space \( R_n \) of dimensions \( k \) and \( l \) are contained in a linear manifold of dimension \( \leq k + l + 1 \).

1344. Prove that if two linear manifolds \( P \) of dimension \( k \) and \( Q \) of dimension \( l \) of the space \( R_n \) have a common point, and \( k + l \geq n \), then their intersection is a linear manifold of dimension \( \geq k + l - n \). What theorems follow from this for three-dimensional and four-dimensional space?

1345. Describe all cases of mutual positions of two planes \( x = a_0 + a_1 t + a_2 s \) and \( x = b_0 + b_1 t + b_2 s \) in \( n \)-dimensional space and indicate the necessary and sufficient conditions for each of these cases.

1346. Suppose \( a_0, a_1, \ldots, a_h \) are any \( k + 1 \) vectors of the space \( R_n \). Prove that all vectors of the form

\[
x = a_0 a_0 + a_1 a_1 + \ldots + a_h a_h
\]

(2)

where the numbers \( a_0, a_1, \ldots, a_h \) satisfy the condition

\[
a_0 + a_1 + \ldots + a_h = 1
\]

(3)

form a linear manifold \( P \) whose dimension is equal to the rank of the set of vectors

\[
a_1 - a_0, \ldots, a_h - a_0.
\]

(4)

\( P \) is a manifold of smallest dimension that contains all of these vectors. Conversely, for any \( k \)-dimensional
called the scalar product of the vectors, provided the following conditions hold:

(a) in the case of a Euclidean space:

\[ (x, y) = (y, x), \]
\[ (x_1 + x_2, y) = (x_1, y) + (x_2, y), \]
\[ (ax, y) = a(x, y) \]

for any real number \( a \).

If \( x \neq 0 \), then \( (x, x) > 0 \);

(b) in the case of a unitary space:

\[ (x, y) = \overline{(y, x)}, \]
\[ (x_1 + x_2, y) = (x_1, y) + (x_2, y), \]

which coincides with (2);

\[ (ax, y) = a(x, y) \]

for any complex number \( a \).

If \( x \neq 0 \), then \( (x, x) > 0 \),

which coincides with (4).

The basis (or, generally, any set of vectors) \( e_1, e_2, \ldots, e_n \) is said to be orthonormal if

\[ (e_i, e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \]

unless otherwise stated, the coordinates of the vectors are assumed to be taken in an orthonormal basis.

The vectors \( x \) and \( y \) are said to be orthogonal if \( (x, y) = 0 \).

The process of orthogonalization of a set of vectors \( a_1, a_2, \ldots, a_n \) is a transition from that set to a new set \( b_1, b_2, \ldots, b_n \), constructed as follows: \( b_1 = a_1 \); \( b_n = a_n \),

\[ \sum_{k=1}^{s} c_i b_k, \quad (k = 2, 3, \ldots, s), \]

where \( c_i = \frac{(a_k, b_i)}{(b_i, b_i)} \) \((i = 1, 2, \ldots, n - 1)\) if \( b_i \neq 0 \) and \( c_i \) is any number if \( b_i = 0 \).

The value of \( c_i \) is obtained by multiplying into \( b_i \) the equation expressing \( b_k \) in terms of \( a_k \) and \( b_i \) \((i = 1, 2, \ldots, n)\), provided that \( b_i \neq 0 \).

1351. Prove that the following properties flow from the properties of a scalar product as indicated in the introduction:

(a) \( (x, y_1 + y_2) = (x, y_1) + (x, y_2) \) for any vectors of a Euclidean (or unitary) space;

(b) \( (x, ay) = a(x, y) \) for any vectors \( x, y \) of a Euclidean space and any real number \( a \);

(c) \( (x, ay) = x(ay) \) for any vectors \( x, y \) of unitary space and any complex number \( a \);

(d) \( (x_1 - x_2, y) = (x_1, y) - (x_2, y) \);

(e) \( (x, 0) = 0 \).

1352. What properties must the bilinear form

\[ g = \sum_{i,j=1}^{n} a_{ij} x_i y_j \]

have so that its value in the coordinates of any two vectors

\[ x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n) \]

of a real vector space \( R^n \) relative to some basis \( e_1, e_2, \ldots, e_n \) may be taken for the scalar product of these vectors, which product defines an \( n \)-dimensional Euclidean space? Find the scalar products of the vectors of the chosen basis.

1353. Given a Hermitian bilinear form \( g = \sum_{i,j=1}^{n} a_{ij} \overline{x_i y_j} \).

The bar over the unknown \( y_j \) means that when \( y_j \) is replaced by its numerical value \( y_j \), \( y_j \) should be replaced by the complex conjugate value \( \overline{y_j} \). Let the matrix \( A = (a_{ij})_n \) of this form be Hermitian, that is, \( a_{ij} = \overline{a_{ji}} \) \((i, j = 1, 2, \ldots, n)\). Show that the values of the corresponding Hermitian quadratic form \( f = \sum_{i,j=1}^{n} a_{ij} \overline{x_i x_j} \)

for arbitrary complex values \( x_1, x_2, \ldots, x_n \) are real, and if the form \( f \) is positive definite, that is, \( f > 0 \) for any complex values \( x_1, x_2, \ldots, x_n \) not all zero, then specifying the scalar product by the equation \( (x, y) = \sum_{i,j=1}^{n} a_{ij} \overline{x_i y_j} \),

where \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) are the coordinates of the vectors \( x \) and \( y \) respectively in some basis \( e_1, \ldots, e_n \).
of the complex vector space \( \mathbb{R}_n \), transforms this space into a unitary space; note that any unitary space may be obtained in this fashion.

1354. Prove that the scalar product of any two vectors 
\[ x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n) \]
of a Euclidean space is expressed by the equation
\[ (x, y) = x_1y_1 + x_2y_2 + \cdots + x_ny_n \]
if and only if the basis in which the coordinates are taken is an orthonormal basis.

1355. Suppose \( L_1 \) and \( L_2 \) are linear subspaces of a Euclidean (or unitary) space \( \mathbb{R}_n \), the dimension of \( L_1 \) being less than that of \( L_2 \); prove that there is a nonzero vector in \( L_2 \) orthogonal to all vectors in \( L_1 \).

1356. Prove that any set of pairwise orthogonal nonzero vectors (in particular, any orthonormal set) is linearly independent.

Verify that the vectors of the following sets are pairwise orthogonal and complete them to form orthogonal bases:

1357. (1, 2, 2, 3), 1358. (1, 1, 1, 0), (2, 3, 2, 4), (1, 2, 3, -3).

Find the vectors that complete the following sets of vectors to form orthonormal bases:

1359. \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right) \),

1360. \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{7}} \right) \),

(\( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \)).

Apply orthogonalization (see Sec. 17, introduction) to construct an orthogonal basis of the subspace spanned by the given set of vectors:

1361. (1, 2, 2, 1), 1362. (1, 1, -1, -2), (1, 1, 3, 3), (0, 1, 2, 3), (0, 2, 3, 7), (1, 3, 3, 8).

1363. (2, 1, 3, -1), (7, 4, 3, -3), (1, 1, -6, 0), (5, 7, 7, 8).

1364. The orthogonal complement of the subspace \( L \) of \( \mathbb{R}_n \) is the set \( L^\perp \) of all vectors in \( \mathbb{R}_n \), each of which is orthogonal to all vectors in \( L \).

Prove that

(a) \( L^\perp \) is a linear subspace of the space \( \mathbb{R}_n \);
(b) the sum of the dimensions of \( L \) and \( L^\perp \) is equal to \( n \);
(c) the space \( \mathbb{R}_n \) is a direct sum of the subspaces \( L \) and \( L^\perp \);
1365. Prove that the orthogonal complement of a linear subspace of the space \( \mathbb{R}_n \) has the following properties:

(a) \( (L^\perp)^\perp = L \),
(b) \( (L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp \),
(c) \( (L_1 \cap L_2)^\perp = L_1^\perp + L_2^\perp \),
(d) \( L^\perp \cap O^\perp = O \).

Here, \( O \) is the zero subspace containing the sole zero vector.

1366. Find the basis of the orthogonal complement \( L^\perp \) of the subspace \( L \) spanned by the vectors:
\[ a_1 = (1, 0, 2, 1), \quad a_2 = (2, 1, 2, 3), \quad a_3 = (0, 1, -2, 0). \]

1367. A linear subspace \( L \) is given by the following equations:
\[ 2x_1 + x_2 + 3x_3 - x_4 = 0, \quad 3x_1 + 2x_3 = 0, \quad 3x_1 + x_2 + 0x_3 - x_4 = 0. \]

Find the equations that specify the orthogonal complement \( L^\perp \).

1368. Show that specification of the linear subspace \( L \) of \( \mathbb{R}_n \) and its orthogonal complement \( L^\perp \) in an orthonormal basis are related to a matrix of the coefficients of a linearly independent system of linear equations whose solution vector is the basis of the other subspace.

1369. Let \( L \) be a linear subspace of the space \( \mathbb{R}_n \). Prove that any vector \( x \) in \( \mathbb{R}_n \) can be uniquely expressed as \( x = y + z \), where \( y \) belongs to \( L \) and \( z \) belongs to \( L^\perp \). \( y \) is termed the orthogonal projection of the vector \( x \) onto the subspace \( L \), and \( z \) is the orthogonal component of \( x \) relative to \( L \). Indicate a procedure for computing \( y \) and \( z \).
Find the orthogonal projection \( y \) and the orthogonal component \( z \) of the vector \( x \) on the linear subspace \( L \):

1370. \( x = (4, -4, -3, 4) \). \( L \) is spanned by the vectors 
\[ a_1 = (1, 1, 1, 1), \quad a_2 = (1, 2, 2, -1), \quad a_3 = (1, 0, 0, 3). \]

1371. \( x = (5, 2, -2, 2) \). \( L \) is spanned by the vectors 
\[ a_1 = (2, 1, 1, -1), \quad a_2 = (1, 1, 3, 0), \quad a_3 = (1, 2, 8, 1). \]

1372. \( x = (7, -4, -1, 2) \). \( L \) is given by the system of equations:
\[
\begin{align*}
2x_1 + x_2 + x_3 + 3x_4 &= 0, \\
3x_1 + 2x_2 + 2x_3 + x_4 &= 0, \\
x_1 + 2x_2 + 2x_3 - 9x_4 &= 0.
\end{align*}
\]

1373. The distance from the point given by the vector \( x \) to the linear manifold \( P = L + x \) is the least of the distances from the given point to the points of the manifold, that is, the minimum of the lengths of the vectors \( x - u \), where \( u \) is a vector of \( P \).

Prove that this distance is equal to the length of the orthogonal component \( z \) of the vector \( x - x_0 \) relative to the linear subspace \( L \), a parallel translation of which yields the manifold \( P \).

1374. Find the distance from the point specified by the vector \( x \) up to the linear manifold specified by the following system of equations:

(a) \( x = (1, 2, -5, 1) \);
\[
\begin{align*}
2x_1 - 2x_2 + x_3 + 3x_4 &= 9, \\
2x_1 - 4x_2 + 2x_3 + 3x_4 &= 12.
\end{align*}
\]

(b) \( x = (2, 4, -4, 2) \);
\[
\begin{align*}
x_1 + 2x_2 + x_3 - x_4 &= 1, \\
x_1 + 3x_2 + x_3 - 3x_4 &= 2.
\end{align*}
\]

1375. Prove that the distance \( d \) from the point given by the vector \( x \) up to the linear manifold \( P = L + x_0 \), where \( L \) is a linear subspace with basis \( a_1, a_2, \ldots, a_k \), is computed with the aid of the Gram determinant (see problem 1415) via the formula
\[
d^2 = \frac{G(a_1, a_2, \ldots, a_k, x - x_0)}{G(a_1, a_2, \ldots, a_k)}.
\]

*1376. The distance between two linear manifolds \( P_1 = L_1 + x_1 \) and \( P_2 = L_2 + x_2 \) is the minimum of the distances of any two points, one of which belongs to \( P_1 \) and the other to \( P_2 \). Prove that this distance is equal to the length of the orthogonal component of the vector \( x_1 - x_2 \) relative to the linear subspace \( L = L_1 + L_2 \).

1377. Find the distance between the two planes \( 2x_1 - 2x_2 + x_3 + x_4 = 0 \) and \( a_1 = (1, 2, 2, 2), \quad a_2 = (2, -2, 1, 2), \quad a_3 = (2, 0, 2, 1), \quad a_4 = (1, -2, 0, -1); \quad x_1 = (4, 5, 3, 2), \quad x_2 = (1, -2, 1, -3).

*1378. A regular \( n \)-dimensional simplex of the Euclidean space \( R_p \) (\( p \geq n \)) is a convex closure (see problem 1347) of a set of equally spaced points \( n + 1 \). The points of the set are termed vertices; the line segments joining them are called edges, and the convex closures of the subsets of \( k + 1 \) points of the given set are termed \( k \)-dimensional faces of the simplex. Two faces are said to be opposite faces if they do not have any vertices in common and any one of the \( n + 1 \) vertices of the simplex is a vertex of one of the faces.

Find the distance between two opposite faces of dimensions \( k \) and \( n - k - 1 \) of an \( n \)-dimensional simplex with edge length unity, and prove that it is equal to the distance between the centres of the faces.

*1379. Let \( e \) be a vector of unit length of the Euclidean (or unitary) space \( R_n \). Prove that any vector \( x \) in \( R_n \) can be uniquely represented as \( x = \alpha e + z \), where \( \langle x, e \rangle = 0 \). The number \( \alpha \) is called the projection of the vector \( x \) on the direction of \( e \) and is denoted as \( \text{pr}_e x \).

Prove that

(a) \( \text{pr}_e (x + y) = \text{pr}_e x + \text{pr}_e y \);

(b) \( \text{pr}_e (\lambda x) = \lambda \text{pr}_e x \);

(c) \( \text{pr}_e x = (x, e) \);

(d) for any orthonormal basis \( e_1, \ldots, e_n \) and any vector \( x \), the equation \( x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i \) holds true.

*1380. Let \( e_1, \ldots, e_n \) be an orthonormal set of vectors of the Euclidean space \( R_n \). Prove that for any vector \( x \) in \( R_n \),

*1381. Let \( e_1, \ldots, e_n \) be an orthonormal set of vectors of the Euclidean space \( R_n \). Prove that for any vector \( x \) in \( R_n \),
we have the inequality (Bessel's inequality)

\[ \sum_{i=1}^{n} (pr_i x)^2 \leq |x|^2. \]

And this inequality becomes the Parseval equality for any \( x \) in \( R^n \) when and only when \( k = n \), that is, the set \( e_1, \ldots, e_n \) is an orthonormal basis.

*1381. Prove the Cauchy-Bunyakovsky inequality

\[ (x, y)^2 \leq (x, x) (y, y) \]

for arbitrary vectors \( x \) and \( y \) of Euclidean space, the equals sign occurring if and only if the vectors \( x \) and \( y \) are linearly dependent.

*1382. Prove the Cauchy-Bunyakovsky inequality

\[ (x, y) (y, x) \leq (x, x) (y, y) \]

for arbitrary vectors \( x \) and \( y \) of unitary space, the equals sign occurring if and only if the vectors \( x \) and \( y \) are linearly dependent.

1383. Using the Cauchy-Bunyakovsky inequality, prove the following inequalities:

(a) \( \left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \)

for any real numbers \( a_1, \ldots, a_n, b_1, \ldots, b_n \) (see problem 503);

(b) \( \left| \sum_{i=1}^{n} a_i b_i \right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 \)

for any complex numbers \( a_1, \ldots, a_n, b_1, \ldots, b_n \) (see problem 505).

1384. Specified in an infinite-dimensional vector space of all real functions continuous on the interval \([a, b]\) with the ordinary addition of functions and multiplication of a function by a scalar is the scalar product \( (f, g) = \int_a^b f(x) g(x) \, dx \). Verify the fulfillment of all properties of a scalar product of Euclidean space (see Sec. 17, introduction) and write down the Cauchy-Bunyakovsky inequality.

Find the lengths of the sides and the interior angles of triangles whose vertices are given by their coordinates.

1385. \( A(2, 4, 2, 4, 2); \quad B(6, 4, 4, 4, 2); \quad C(7, 2) \).

1386. \( A(1, 2, 3, 2, 1); \quad B(3, 4, 0, 4, 3); \quad C(1, 1, 1, 1, 1) \).

1387. Prove the following generalization of the theorem of elementary mathematics concerning two perpendicular lines: if the vector \( x \) of a Euclidean (or unitary) space is orthogonal to each of the vectors \( a_1, a_2, \ldots, a_n \), then it is orthogonal to each vector of the linear subspace spanned by the vectors \( a_1, a_2, \ldots, a_n \).

1388. Prove that if \( z = ay \), then \( |x| = |a| \cdot |y| \). Here, \( |x|, |y| \) are the lengths of the vectors \( x \) and \( y \).

*1389. Prove that the square of the diagonal of a rectangular \( n \)-dimensional parallelepiped is equal to the sum of the squares of its edges emanating from a single vertex (this is an \( n \)-dimensional generalization of the Pythagoras theorem).

*1390. Prove a theorem stating that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

1391. Prove a theorem stating that the square of a side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of those sides into the cosine of the angle between them. Do this by using the scalar multiplication of vectors.

1392. Use the Cauchy-Bunyakovsky inequality to prove the triangle inequality: if \( p(X, Y) \) is the distance between the points \( X \) and \( Y \), then for any three points \( A, B \) and \( C \) we have \( p(A, C) \leq p(A, B) + p(B, C) \), equality occurring if and only if the vector drawn from \( A \) to \( B \) and the vector drawn from \( B \) to \( C \) are collinear and in the same direction.

1393. Find the number of diagonals of a \( n \)-dimensional cube, which diagonals are orthogonal to a given diagonal.

1394. Find the length of a diagonal of an \( n \)-dimensional cube with edge \( a \) and find the limit of that length as \( n \to \infty \).

1395. Prove that all diagonals of an \( n \)-dimensional cube

Find the lengths of the sides and the interior angles of triangles whose vertices are given by their coordinates.
form the same angle $q$, with all the edges. Find that angle and also its limit as $n \to \infty$. For what value of $n$ do we obtain $q = 60^\circ$?

1396. Find an expression for the radius $R$ of a sphere described about an $n$-dimensional cube using the edge $a$; which of the quantities $R$ and $a$ is greater for different values of $n$?

1397. Prove that the orthogonal projection of any edge of an $n$-dimensional cube on any diagonal of that cube is equal, in absolute value, to $\frac{1}{n}$ of the length of the diagonal.

1398. Prove that the orthogonal projections of the vertices of an $n$-dimensional cube on any diagonal divides it into $n$ equal parts.

1399. Let $x$ and $y$ be nonzero vectors of the Euclidean space $\mathbb{R}^n$. Prove that

(a) $x = ay$, where $a > 0$ if and only if the angle between $x$ and $y$ is zero;

(b) $x = ay$, where $a < 0$ if and only if the angle between $x$ and $y$ is equal to $\pi$.

1400. Prove that of all vectors of the linear subspace $L$ the smallest angle with the given vector $x$ is formed by the orthogonal projection $y$ of the vector $x$ on $L$. Here, the equality $\cos (x, y) = \cos (x', y)$, where $y' \in L$, holds if and only if $y' = ay$, where $a > 0$.

1401. Find the angle between the diagonal of an $n$-dimensional cube and its $k$-dimensional face.

Find the angle between the vector $x$ and a linear subspace spanned by the vectors $a_1, a_2, \ldots, a_k$:

1402. $x = (2, 2, 1, 4, 1)$;

$a_1 = (3, 3, 3, 1, -1)$;

$a_2 = (4, 3, 2, 4, 3)$;

$a_3 = (3, 4, 3, 2, 4)$;

$a_4 = (2, 1, 4, 3, 2)$;

1403. $x = (1, 0, 3, 0, 0)$;

$a_1 = (3, 0, 4, 3, 0)$;

$a_2 = (1, 4, 3, 0)$;

$a_3 = (4, 3, 2, 4, 3)$;

1404. The inner angle between two linear subspaces $L_1$ and $L_2$ of the linear space $\mathbb{R}^n$, which subspaces do not have any common nonzero vectors, is used for the minimum of the angles between the nonzero vectors $x_1$ and $x_2$, where $x_1 \in L_1$, $x_2 \in L_2$. If the intersection $L_1 \cap L_2 = D \neq 0$, and $D \neq L_1$, $D \neq L_2$, then the angle between $L_1$ and $L_2$ is the angle between their intersections with the orthogonal complement $D^*$ to the intersection $D$. If one of the given subspaces is contained in the other (in particular, if they coincide), then the angle between them is assumed to be zero. The angle between linear manifolds is the angle between the subspaces that correspond to them. Show that the angle between any subspaces or manifolds is always defined and is zero if and only if one of the subspaces or manifolds is contained in the other or the manifolds are parallel.

1405. Find the angle between the two-dimensional faces $A_0A_1A_2$ and $A_0A_3A_4$ of a regular four-dimensional simplex (see problem 1378) $A_0A_1A_2A_3A_4$.

1406. Find the angle between the planes $a_0 + a_1t_1 + a_2t_2$ and $b_0 + b_1t_1 + b_2t_2$, where $a_0 = (3, 1, 0, 1)$, $a_1 = (1, 0, 0, 0)$, $a_2 = (0, 1, 0, 0)$, $b_0 = (2, 1, 1, 3)$, $b_1 = (1, 1, 1, 1)$, $b_2 = (1, -1, 1, -1)$.

1407. Given a linearly independent set of vectors $e_1, e_2, \ldots, e_s$ and two orthogonal sets of nonzero vectors $f_1, f_2, \ldots, f_s$ and $g_1, g_2, \ldots, g_s$ such that the vectors $f_k$ and $g_k$ are linearly expressible in terms of $e_1, e_2, \ldots, e_s$ ($k = 1, 2, \ldots, s$). Prove that $f_k = g_k$ ($k = 1, 2, \ldots, s$), where $\alpha_k \neq 0$.

1408. Let $\mathbb{R}_{n+1}$ be a Euclidean space, the vectors of which are all polynomials of degree $\leq n$ in the sole unknown $x$ with real coefficients, and the scalar product of the polynomials $f(x)$ and $g(x)$ is defined thus:

$$(i, g) = \int_{-1}^{+1} f(x) g(x) \, dx.$$ 

Prove that the following polynomials (called Legendre polynomials)

$P_0(x) = 1, \quad P_k(x) = \frac{1}{2^{k-1} k!} \int_{-1}^{+1} (x^2 - 1)^k \, dx$ 

form an orthogonal basis of the space $\mathbb{R}_{n+1}$.

1409. Using the definition of a Legendre polynomial given in the preceding problem, find the polynomials $P_k(x)$
for \( k = 0, 1, 2, 3, 4 \). Convince yourself that \( P_k(x) \) is of degree \( k \) and write out in full the expression for \( P_k(x) \) in powers of \( x \) for arbitrary \( k \\

*1410. Compute the "length" of the Legendre polynomial \( P_k(x) \) as a vector of the Euclidean space \( R_{n+1} \) of problem 1408.

*1411. Compute the value of the Legendre polynomial \( P_k(x) \) for \( x = 1 \).

*1412. Give prove that if the orthogonalization process is applied to the basis \( 1, x, x^2, \ldots, x^n \) of the Euclidean space \( R_{n+1} \) of problem 1408, the result will be the polynomials \( p_0(x), p_1(x), \ldots, p_n(x) \) that differ from the corresponding Legendre polynomials by constant factors only. Find those factors.

1413. Suppose the vectors \( a_1, a_2, \ldots, a_n \) are carried into the vectors \( b_1, b_2, \ldots, b_n \) respectively by orthogonalization. Prove that \( b_k \) is an orthogonal component of the vector \( a_k \) relative to the linear subspace \( L_{k-1} \) spanned by \( a_1, \ldots, a_{k-1} \) \((k > 1)\). Furthermore, prove that

\[
0 \leq |b_k| = |a_k| \quad (k = 1, 2, \ldots, n).
\]

Here, \( |b_k| = 0 \) if and only if \( a_k \) is linearly expressible in terms of \( a_1, \ldots, a_{k-1} \) \((k > 1)\) or \( a_1 = 0 \) \((k = 1)\); \( |b_k| = |a_k| \) if and only if \( (a_k, a_j) = 0 \) \((j = 1, 2, \ldots, k - 1; k > 1)\) or \( k = 1 \).

*1414. Prove that the integral \( \int \left[ f(x) \right]^2 \, dx \) where \( f(x) \) is an \( n \)-th-degree polynomial with real coefficients and leading coefficient unity, attains its minimum, equal to \( \frac{2^n}{(2n+1)P_n(n)^2} \), if and only if \( f(x) = \frac{2^n}{(2n+1)P_n(n)} \), where \( P_n(x) \) is a Legendre polynomial of degree \( n \) (see problem 1408).

1415. The Gram determinant of the vectors \( a_1, a_2, \ldots, a_k \) of the Euclidean (or unitary) space \( R_n \) is the determinant

\[
\begin{vmatrix}
(a_1, a_1) & (a_1, a_2) & \cdots & (a_1, a_k) \\
(a_2, a_1) & (a_2, a_2) & \cdots & (a_2, a_k) \\
\vdots & \vdots & \ddots & \vdots \\
(a_k, a_1) & (a_k, a_2) & \cdots & (a_k, a_k)
\end{vmatrix}
\]

Prove that the Gram determinant does not change the orthogonalization is applied to the vectors \( a_1, \ldots, a_k \) that is to say, if after orthogonalization the vector \( a_1, \ldots, a_k \) are carried into the vectors \( b_1, \ldots, b_k \), then

\[
g(a_1, \ldots, a_k) = g(b_1, \ldots, b_k)
\]

Using this fact, determine the geometric meaning of \( g(a_1, a_2) \) and \( g(a_1, a_2, a_3) \) on the assumption that the vectors are linearly independent.

*1416. Prove that for the vectors \( a_1, \ldots, a_k \) of a Euclidean (or unitary) space to be linearly dependent, it is necessary and sufficient for the Gram determinant of these vectors to be zero.

*1417. Two bases \( e_1, \ldots, e_n \) and \( f_1, \ldots, f_n \) of a Euclidean (or unitary) space are said to be reciprocal if

\[
(e_i, f_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}
\]

Prove that for any basis \( e_1, \ldots, e_n \) the reciprocal basis exists and is uniquely defined.

1418. Let \( S \) be the transition matrix from the basis \( e_1, \ldots, e_n \) to the basis \( e'_1, \ldots, e'_n \). Find the transition matrix \( T \) from the basis \( f_1, \ldots, f_n \), that is reciprocal to the basis \( e'_1, \ldots, e'_n \), to the basis \( f'_1, \ldots, f'_n \), that is reciprocal to the basis \( e_1, \ldots, e_n \).

*1419. Prove that the Gram determinant \( g(a_1, \ldots, a_n) \) is equal to zero if the vectors \( a_1, \ldots, a_k \) are linearly dependent, and is positive if they are linearly independent.

1420. Prove that if the linearly independent vectors \( a_1, \ldots, a_n \) are carried into the vectors \( b_1, \ldots, b_n \) by orthogonalization, then \( \sum_{k=1}^{n} |b_k|^2 = \frac{2^n}{(2n+1)P_n(n)^2} \), \((k = 1, 2, \ldots, n)\); (the Gram determinant with a zero number of vectors is assumed equal to unity).

*1421. In the space of polynomials of degree not exceeding \( n \) in a single unknown \( x \) with real coefficients, the scalar product is given by the equation \( (f, g) = \int_{-1}^{1} f(x) g(x) \, dx \).
Find the distance from the coordinate origin to the linear manifold consisting of all \( n \)-th degree polynomials with leading coefficient unity.

1422. Prove that the following equation holds true for the Gram determinant:

\[
0 \leq g(a_1, \ldots, a_k) \leq |a_1|^2 \ldots |a_k|^2.
\]

Here, \( g(a_1, \ldots, a_k) = 0 \) if and only if the vectors \( a_1, \ldots, a_k \) are linearly dependent, and \( g(a_1, \ldots, a_k) = \sum a_i \ldots a_k \) if and only if the vectors \( a_1, \ldots, a_k \) are pairwise orthogonal or at least one of them is zero.

1423. Use the preceding problem to prove the Hadamard inequality, namely, that if \( D = \begin{vmatrix} a_{ij} \end{vmatrix} \) is a determinant with real elements, then \( D^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \) (see problem 923) and equality occurs if and only if either

\[
\sum_{k=1}^{n} a_{ik}a_{jk} = 0 \quad (i \neq j; i, j = 1, 2, \ldots, n)
\]
or the determinant \( D \) contains a zero row. How is the assertion altered if the determinant has complex elements?

1424. Prove that the determinant \( D_f \) of a positive definite quadratic form \( f = \sum_{i,j=1}^{n} a_{ij}x_i x_j \) satisfies the inequality \( D_f \leq \prod_{i,j=1}^{n} a_{ij} \).

1425. Prove that any real symmetric matrix \( A = (a_{ij}) \) with nonnegative principal minors is a Gram matrix, that is, there exists a set of vectors \( e_1, e_n \) of the Euclidean space \( \mathbb{R}^n \) such that \( (e_i, e_j) = a_{ij} \) \((i, j = 1, 2, \ldots, n)\).

1426. Prove that any Hermitian matrix \( A = (a_{ij}) \) with nonnegative principal minors is a Gram matrix, that is, there exists a set of vectors \( e_1, \ldots, e_n \) of the unitary space \( \mathbb{C}^n \) such that

\[
(e_i, e_j) = a_{ij} \quad (i, j = 1, 2, \ldots, n).
\]

1427. Let us determine inductively, by the following procedure, the volume of an \( n \)-dimensional parallelepiped spanned by the linearly independent vectors \( a_1, a_2, \ldots, a_n \) of a Euclidean space:

\[
D = |a_1|.|a_2| \ldots |a_n|.
\]

\[
V(a_1, \ldots, a_n) = \sqrt{|a_1|.|a_2| \ldots |a_n|} = |D|,
\]

where \( D \) is a determinant of the coordinates of the given vectors in some orthonormal basis of \( n \)-dimensional space.

1428. Prove that the volume of a parallelepiped does not exceed the product of the length of its edges emanating from a single vertex, and is equal to that product if and only if the edges are pairwise orthogonal, that is, if the parallelepiped is rectangular.

1429. Prove the following property of a Gram determinant:

\[
g(a_1, \ldots, a_k, b_1, \ldots, b_l) \leq g(a_1, \ldots, a_k)g(b_1, \ldots, b_l),
\]

equality occurring if and only if

\[
(a_i, b_j) = 0 \quad (i = 1, 2, \ldots, k; j = 1, 2, \ldots, l)
\]
or if at least one of the subsets \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_l \) is linearly dependent.

1430. Prove the following property of the volume of a parallelepiped: \( V(a_1, \ldots, a_k; b_1, \ldots, b_l) \leq V(a_1, \ldots, a_k) \) \( V(b_1, \ldots, b_l) \); note that equality occurs if and only if \( (a_i, b_j) = 0 \) \((i = 1, 2, \ldots, k; j = 1, 2, \ldots, l)\).

1431. Prove that if \( A \) is a real symmetric matrix of order \( n \) with nonnegative principal minors, \( A_1 \) is a matrix of order \( k < n \) in the upper left-hand corner, and \( A_2 \) is a matrix of order \( n - k \) in the lower right-hand corner of \( A \), then \( |A| \leq |A_1|.|A_2| \) (compare with problem 922).

1432. Solve the same problem as in *1431 with the proviso that \( A \) is a Hermitian matrix with nonnegative principal minors.

1433. Find the conditions that are necessary and sufficient for \( C_n^{k+1} \) positive numbers

\[
a_{ij} \quad (i, j = 0, 1, 2, \ldots, n; i > j)
\]

to be (a) the distances of all possible pairs of vertices of some \( n \)-dimensional simplex of the Euclidean space \( \mathbb{R}^n \).
a set of \( n + 1 \) points not lying in an \((n - 1)\)-dimensional linear manifold); 
(b) the distances of all possible pairs of points of some set of \( n + 1 \) points in the Euclidean space \( R_n \).

Sec. 18. Linear Transformations of Arbitrary Vector Spaces

In this section we are concerned mainly with linear transformations of affine vector spaces. The transformations of Euclidean and unitary spaces are considered in the next section.

Linear transformations are denoted by \( \varphi, \psi \), and so on, the image of the vector \( x \) under the transformation \( \varphi \) by \( \varphi x \); the set of vectors \( \varphi a_1, \ldots, \varphi a_n \) is denoted by \( \varphi (a_1, \ldots, a_n) \).

The matrix of a linear transformation \( \varphi \) in the basis \( e_1, \ldots, e_n \) is the matrix \( A_\varphi \), whose columns are made up of the coordinates of the images of the basis \( \varphi e_1, \ldots, \varphi e_n \) in the same basis \( e_1, \ldots, e_n \); in other words, the matrix \( A_\varphi \) is defined by the equation

\[
\varphi (e_1, \ldots, e_n) = (e_1, \ldots, e_n) A_\varphi. \tag{1}
\]

Let \( T \) be the transition matrix from the basis \( e_1, \ldots, e_n \) to the basis \( f_1, \ldots, f_n \) (see introduction to Sec. 16), and let \( A_\varphi \) and \( B_\psi \) be matrices of the transformation \( \varphi \) in the first and second bases respectively. We then have the relation

\[
B_\psi = T^{-1} A_\varphi T. \tag{2}
\]

The coordinates \( y_1, \ldots, y_n \) of the image \( \varphi x \) of the vector \( x \) under a linear transformation \( \varphi \) are expressed in terms of the coordinates \( x_1, \ldots, x_n \) of the preimage \( x \) in the same basis by means of the matrix \( A_\varphi = (a_{ij}) \) of the linear transformation \( \varphi \) in the same basis in the following manner: \( y_i = \sum_{j=1}^{n} a_{ij} x_j \) \( (i = 1, \ldots, n) \) or, in matrix form,

\[
\begin{pmatrix}
y_1 \\
\vdots \\
y_n
\end{pmatrix} = A_\varphi
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}. \tag{3}
\]

The sum \( \varphi + \psi \) and the product \( \varphi \psi \) of two linear transformations \( \varphi \) and \( \psi \), and the product \( \alpha \varphi \) of the scalar \( \alpha \) by the linear transformation \( \varphi \) of the space \( R_n \) are transformation defined as follows:

\[
(\varphi \psi) x = \varphi (\psi x), \quad (\varphi + \psi) x = \varphi x + \psi x,
\]
for any vector \( x \) in the space \( R_n \).

1434. Prove that the rotation of a plane through angle \( \omega \) about the origin of coordinates is a linear transformation, and find the matrix of that transformation in an orthonormal basis if the positive direction for reckoning angles coincides with the smallest rotation that carries the first basis vector into the second.

1435. Prove that the rotation of three-dimensional space through an angle \( \frac{2\pi}{3} \) about a straight line given in a rectangular coordinate system by the equations \( x_1 = x_2 = x_3 \) is a linear transformation, and find the matrix of that transformation relative to the basis of unit vectors \( e_1, e_2, e_3 \) of the coordinate axes.

1436. Prove that projecting three-dimensional space on the coordinate axis of the vector \( e_1 \) parallel to the coordinate plane of vectors \( e_2 \) and \( e_3 \) is a linear transformation; find its matrix relative to the basis \( e_1, e_2, e_3 \).  

1437. Prove that projecting three-dimensional space on the coordinate plane of the vectors \( e_1, e_2 \) parallel to the coordinate axis of the vector \( e_3 \) is a linear transformation; find its matrix relative to the basis \( e_1, e_2, e_3 \).

1438. Prove that the orthogonal projecting of three-dimensional space on the axis forming equal angles with the axes of a rectangular coordinate system is a linear transformation; find its matrix relative to the basis of unit vectors of the coordinate axes.

1439. Suppose the space \( R_n \) is a direct sum of linear subspaces \( L_1 \) with basis \( a_1, \ldots, a_n \) and \( L_2 \) with basis \( a_{n+1}, \ldots, a_n \). Prove that the projection of the space on \( L_1 \) parallel to \( L_2 \) is a linear transformation; find the matrix of the transformation relative to the basis \( a_1, \ldots, a_n \).

1440. Prove that there exists a unique linear transformation of the space \( R_n \) that carries the given linearly independent vectors \( a_1, \ldots, a_n \) into the given vectors \( b_1, \ldots, b_n \). The problem is to find the matrix of that transformation relative to the basis \( a_1, \ldots, a_n \).

Determine which of the following transformations is specified by giving the coordinates of the vector \( \varphi x \).
functions of the coordinates of the vector \( x \), are linear; in the case of linearity, find their matrices in the same basis in which the coordinates of the vectors \( x \) and \( qw \) are given.

1441. \( qw = (x_1, x_2, x_3, x_4, x_5) \).

1442. \( qw = (x_1, x_2, 1, x_3, |2|). \)

1443. \( qw = (2x_1 + x_2, x_1 + x_2, x_3). \)

1444. \( qw = (x_1 - x_2 + x_3, x_2, x_3). \)

Prove that there exists a unique linear transformation of three-dimensional space that carries the vectors \( a_1, a_2, a_3 \) into the vectors \( b_1, b_2, b_3 \) respectively, and find the matrix of the transformation relative to the same basis in which the coordinates of all vectors are given:

1445. \( a_1 = (1, 3, 5), b_1 = (1, 1, 1), \)

\( a_2 = (0, 1, 2), b_2 = (1, 1, -1), \)

\( a_3 = (1, 0, 0), b_3 = (2, 1, 2). \)

1446. \( a_1 = (2, 0, 3), b_1 = (1, 2, -4), \)

\( a_2 = (1, 1, 2), b_2 = (1, 5, -2), \)

\( a_3 = (3, 1, 2), b_3 = (1, 1, 1). \)

1447. Suppose the linear transformation \( q \) of the space \( R^n \) carries the linearly independent vectors \( a_1, \ldots, a_n \) into the vectors \( b_1, \ldots, b_n \) respectively. Prove that the matrix \( A \) of that transformation in some basis \( e_1, \ldots, e_n \) can be found from the equation \( A = B \Lambda B^{-1} \), where the columns of the matrices \( A \) and \( B \) consist of the coordinates of the vectors \( a_1, \ldots, a_n \) and, respectively, \( b_1, \ldots, b_n \) relative to the basis \( e_1, \ldots, e_n \).

1448. Prove that the transformation of three-dimensional space \( qw = w \cdot a \cdot a \) where \( a = (1, 2, 3) \) is a linear transformation, and find the matrices in the orthonormal basis \( t, x, z \) in which the coordinates of all vectors are given, and in the basis \( b_1 = (1, 0, 1), b_2 = (0, 1, 1), b_3 = (1, 1, 1). \)

1449. Show that multiplication of second order square matrices, both on the left and on the right by a given matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), constitutes linear transformations of the space of all second order matrices; find the matrices of the transformations relative to a basis consisting of the matrices

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

1450. Demonstrate that differentiation is a linear transformation of the space of all polynomials of degree \( n \) in one unknown with real coefficients.

Find the matrix of that transformation in the basis:

(a) \( x, x^2, \ldots, x^n; \)

(b) \( x - x^n, \frac{(x - x)^2}{2!}, \ldots, \frac{(x - x)^n}{n!} \), where \( c \) is a real number.

1451. What change will the matrix of a linear transformation undergo if two vectors \( e_i, e_j \) are interchanged in the basis \( e_1, e_2, \ldots, e_n? \)

1452. A linear transformation \( q \) relative to the basis \( e_1, e_2, e_3, e_4 \) has the matrix

\[
\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}.
\]

Find the matrix of the transformation relative to the basis:

(a) \( e_1, e_2, e_3, e_4; \)

(b) \( e_1 - e_2, e_1 + e_2, e_3 + e_4, e_1 + e_2 + e_3 + e_4. \)

1453. The linear transformation \( q \) has, relative to the basis \( e_1, e_2, e_3 \), the matrix

\[
\begin{pmatrix} \frac{15}{8} & -\frac{11}{2} & 5 \\ 0 & -\frac{15}{2} & 8 \\ \frac{15}{8} & -\frac{11}{2} & 6 \end{pmatrix}.
\]

Find the matrix relative to the basis

\( f_1 = 2e_1 + 3e_2, f_2 = 3e_1 - 4e_2 + e_3, f_3 = e_1 + 2e_2 + 2e_3. \)

1454. The linear transformation \( q \) has, relative to the basis

\( a_1 = (8, 6, 7), a_2 = (-16, 5, 13), a_3 = (5, 3, 3). \)

\[
\begin{pmatrix} 15 & -11 & 5 \\ 0 & -15 & 8 \\ 8 & -6 & 6 \end{pmatrix}.
\]
Find the matrix relative to the bases

\[ b_1 = (1, -2, 1), \quad b_2 = (3, -1, 2), \quad b_3 = (2, 1, 2). \]

1455. Prove that the matrices of one and the same linear transformation in two bases coincide if and only if the transition matrix from one of the bases to the other coincides with the matrix of that linear transformation relative to one of the given bases.

1456. Prove that any linear transformation \( \phi \) of one dimensional space reduces to the multiplication of all vectors by the same scalar, that is, \( \phi x = ax \) for any vector \( x \).

1457. Let the transformation \( \phi \) have the matrix \( \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \) relative to the basis \( a_1 = (1, 2), \quad a_2 = (2, 3) \). The transformation \( \psi \) has the matrix \( \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \) relative to the basis \( b_1 = (3, 1), \quad b_2 = (1, 2). \)

Find the matrix of the transformation \( \psi \mid \phi \) relative to the bases \( b_1, b_2 \).

1458. The transformation \( \psi \) has the matrix \( \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix} \) relative to the basis \( a_1 = (3, 7), \quad a_2 = (1, 2) \) and the transformation \( \phi \) has the matrix \( \begin{pmatrix} 4 & 3 \\ 6 & 7 \end{pmatrix} \) relative to the basis \( b_1 = (6, 7), \quad b_2 = (5, 6). \)

Find the matrix of the transformation \( \phi \mid \psi \) in the same basis, in which the coordinates of all vectors are given.

1459. Suppose \( \phi \) is a linear transformation of the space of polynomials of degree \( n \) with real coefficients such that it changes each polynomial into its derivative. Show that \( \phi^n = n! \cdot I \).

1460. Let \( \phi \) be a linear transformation of differentiation, and let \( \psi \) be the multiplication by \( x \) in an infinite dimensional space of all polynomials in \( x \) with real coefficients. Prove that \( \phi \psi = n \cdot \psi \phi \).

1461. Show that the linear transformations of addition and multiplication by a scalar of a vector space form a vector space. Find the dimension of this space.

Find the eigenvalues and the eigenvectors of linear transformations specified in some basis by the following matrices:

1462. A linear transformation \( \phi \) of the space \( \mathbb{R}^3 \) to be nonsingular if its matrix \( A \) in some linear basis is nonsingular, that is, \( |A| \neq 0 \). Prove that this definition is equivalent in any one of the following:

(a) \( \phi \) is linear and \( \phi(e) \neq 0 \) for any \( e \in \mathbb{R}^3 \); (b) \( \phi \) is linear and \( \phi(e) \neq 0 \) for any \( e \in \mathbb{R}^3 \) that is a non-zero vector; (c) \( \phi \) is linear and \( \phi(e) \neq 0 \) for any \( e \in \mathbb{R}^3 \) that is a non-zero vector.

1463. Let \( x \) be an eigenvector of the linear transformation \( \phi \) that belongs to the eigenvalue \( \lambda \), and let \( f(\lambda) \) be a polynomial. Prove that the same vector \( x \) will be an eigenvector of the transformation \( \phi' \) that belongs to the eigenvalue \( f(\lambda) \). In other words, prove that \( \phi(x) = \lambda x \) implies \( \phi'(x) = f(\lambda)x \).

1464. Let \( x \) be an eigenvector of the linear transformation \( \phi \) belonging to the eigenvalue \( \lambda \), and let \( f(\lambda) \) be a function for which the transformation \( f(\phi) \) is meaningful in some basis. If \( \phi \) has a matrix \( A \), then \( f(\phi) \) is defined in some basis by the matrix \( f(A) \), and it can be proved that \( f(\phi) \) does not depend on the choice of basis. Prove that the same vector \( x \) will be an eigenvector of the transformation \( f(\phi) \) which eigenvector is associated with the eigenvalue \( f(\lambda) \).

Find the eigenvalues and the eigenvectors of linear transformations specified in some basis by the following matrices:

1465. \[
\begin{pmatrix}
2 & 1 & 2 \\
5 & -3 & 3 \\
1 & 0 & -2
\end{pmatrix}
\]

1466. \[
\begin{pmatrix}
0 & 1 & 0 \\
-4 & 4 & 0 \\
3 & 0 & 2
\end{pmatrix}
\]

1467. \[
\begin{pmatrix}
4 & 5 & 2 \\
3 & 7 & 1 \\
6 & 9 & 4
\end{pmatrix}
\]

1468. \[
\begin{pmatrix}
1 & 3 & 3 \\
2 & -1 & 13 \\
-1 & -3 & 8
\end{pmatrix}
\]

1469. \[
\begin{pmatrix}
1 & 3 & 4 \\
4 & 7 & 3 \\
6 & 7 & 7
\end{pmatrix}
\]

1470. \[
\begin{pmatrix}
7 & -12 & 6 \\
10 & -19 & 10 \\
12 & -24 & 13
\end{pmatrix}
\]
1471. \[
\begin{pmatrix}
4 & 5 & 7 \\
1 & 4 & 9 \\
-4 & 0 & 5
\end{pmatrix}
\]

1472. \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]

1473. \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

1474. \[
\begin{pmatrix}
3 & -1 & 0 \\
1 & 1 & 0 \\
3 & 0 & 5 -3
\end{pmatrix}
\]

1475. Prove that the eigenvectors of a linear transformation that belong to different eigenvalues are linearly independent.

1476. Prove that any square matrix $A$ having distinct eigenvalues is similar to a diagonal matrix (over a field containing both elements of the matrix and its eigenvalues).

1477. Prove that if the linear transformation $\varphi$ of the space $R_n$ has a distinct eigenvalues, then any linear transformation $\psi$ that commutes with $\varphi$ has a basis of eigenvectors, and any eigenvector of $\varphi$ will be an eigenvector of $\psi$.

1478. Prove that the matrix of a linear transformation in some basis is diagonal if and only if the basis consists of the eigenvectors of the given transformation.

Determine which of the following matrices of linear transformations can be reduced to diagonal form by going over to a new basis. Find that basis and the corresponding matrix:

1479. \[
\begin{pmatrix}
1 & 3 & -1 \\
-3 & 1 & -1 \\
3 & 1 & 1
\end{pmatrix}
\]

1480. \[
\begin{pmatrix}
6 & -5 & -3 \\
3 & -2 & -2 \\
2 & -2 & 0
\end{pmatrix}
\]

1481. \[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

1482. \[
\begin{pmatrix}
4 & -3 & 1 & 2 \\
5 & -8 & 5 & 4 \\
6 & 12 & 8 & 5 \\
3 & -3 & 2 & 2
\end{pmatrix}
\]

*1484. For the matrix $A = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix}$

of order $n$, find a nonsingular matrix $T$ for which the matrix $B = T^{-1}AT$ is diagonal, and find $B$.

1485. A minimal polynomial for a vector $x$ relative to a linear transformation $\varphi$ is a polynomial $g_\varphi(\lambda)$ with leading coefficient 1 and having the smallest degree from among all cancelling polynomials for $x$ relative to $\varphi$, that is, the polynomials $f(\lambda)$ with the property that $f(\varphi)x = 0$.

A similar definition is given for the minimal polynomial $g(\lambda)$ relative to a linear transformation $\varphi$ for the whole space. Prove that the minimal polynomial $g(\lambda)$ of the linear transformation $\varphi$ is equal to the least common multiple of the minimal polynomials for vectors of any basis of the space relative to $\varphi$.

*1486. Find the conditions for which a matrix $A$, which is the matrix of a linear transformation has the numbers $a_1, a_2, \ldots, a_n$ on the secondary diagonal and zeros elsewhere, is similar to a diagonal matrix.

1487. Find the eigenvalues and the eigenvectors of a linear transformation which is a differentiation of polynomials of degree $\leq n$ with real coefficients.

1488. Let $\varphi$ be a linear transformation of the space $R_n$. The set of all vectors $\varphi x$, where $x$ is any vector in $R_n$, is termed the image of $R_n$ under the transformation $\varphi$ or the domain of values of $\varphi$. The collection of all vectors $x$ in $R_n$ such that $\varphi x = 0$ is termed the total pre-image of zero under the transformation $\varphi$, or the kernel of $\varphi$. Prove that (a) the domain of values of $\varphi$ is a linear subspace of $R_n$ whose dimension is equal to the rank of $\varphi$; (b) the kernel of $\varphi$ is a linear subspace of the space $R_n$ whose dimension is equal to the nullity of $\varphi$, that is, the difference between $n$ and the rank of $\varphi$.

1489. Let $\varphi$ be a linear transformation and let $L$ be a subspace of $R_n$. Prove that (a) the image $\varphi L$ and (b) the total pre-image $\varphi^{-1}L$ of the subspace $L$ under a linear transformation $\varphi$ are again subspaces.

1490. Prove that for a nonsingular linear transformation $\varphi$ of the space $R_n$, the dimension of its image $\varphi L$ and
(b) the total pre-image \( \varphi^{-1}L \) of any linear subspace \( L \) is equal to the dimension of \( L \).

*1491. Use "\( \dim L \)" to denote the dimension of the linear subspace \( L \) and the notation "nullity \( \varphi \)" to denote the nullity of the linear transformation \( \varphi \). Prove that the dimensions of the image and the total pre-image of the subspace \( L \) of the space \( \mathbb{R}^n \) under the transformation \( \varphi \) satisfy the inequalities

\[
\begin{align*}
(a) \quad \dim L - \text{nullity } \varphi & \leq \dim \varphi L \leq \dim L; \\
(b) \quad \dim L \leq \dim \varphi^{-1}L \leq \dim L + \text{nullity } \varphi.
\end{align*}
\]

*1492. Use the preceding problem to prove the Sylvester inequality for the rank of a product of two square matrices \( A \) and \( B \) of order \( n \): 
\[r_A + r_B - n \leq r_{AB} \leq \min \{r_A, r_B\}\] (see problem 931).

1493. Prove that

\[
\begin{align*}
(a) \quad \text{rank } (\varphi + \psi) & \leq \text{rank } \varphi + \text{rank } \psi; \\
(b) \quad \text{nullity } (\varphi \psi) & \leq \text{nullity } \varphi + \text{nullity } \psi \text{ for any linear transformations } \varphi \text{ and } \psi \text{ of the space } \mathbb{R}^n.
\end{align*}
\]

1494. Find the eigenvalues and the eigenvectors of the linear transformation \( \varphi \) specified in the basis \( a_1, a_2, a_3, a_4 \) by the matrix

\[
\begin{pmatrix}
1 & 0 & 2 & -1 \\
0 & 1 & 4 & -2 \\
2 & -1 & 0 & 1 \\
2 & -1 & -1 & 2
\end{pmatrix}
\]

Show that the subspace spanned by the vectors \( a_1 + 2a_2 \) and \( a_1 + a_2 + 2a_4 \) is invariant under \( \varphi \).

*1495. Prove that the number of linearly independent eigenvectors of the transformation \( \varphi \) associated with the single eigenvalue \( \lambda_n \) does not exceed the multiplicity of \( \lambda_n \) as a root of the characteristic polynomial of the transformation \( \varphi \).

*1496. Prove that a linear subspace spanned by an arbitrary set of eigenvectors of the transformation \( \varphi \) is invariant under \( \varphi \).

1497. Prove that the set of all eigenvectors of the linear transformation \( \varphi \) associated with one and the same eigenvalue \( \lambda_n \) (together with the zero vector) is a linear space that is invariant under \( \varphi \).

1498. Prove that all nonzero vectors of a space \( \mathbb{R}^n \) under the linear transformation \( \varphi \) if and only if it is a similarity transformation, that is, \( \varphi = A x \) with the same \( A \) for any vector \( x \).

1499. Prove that any subspace \( L \) that is invariant under a nonsingular linear transformation \( \varphi \) will also be invariant under the linear transformation \( \varphi^{-1} \).

1500. Prove that (a) the image \( \varphi L \) and (b) the total pre-image \( \varphi^{-1}L \) of the linear subspace \( L \), which is invariant under the linear transformation \( \varphi \), are themselves invariant under \( \varphi \).

1501. Find all the linear subspaces of the space of polynomials in one unknown of degree \( \leq n \) with real coefficients which subspaces are invariant under the transformation \( \varphi \) that carries any polynomial into its derivative.

1502. Prove that the matrix of the linear transformation \( \varphi \) of \( n \)-dimensional space in the basis \( a_1, a_2, \ldots, a_n \) is a semidisintegrated block matrix of the form

\[
\begin{pmatrix}
A_1 & B \\
0 & A_2
\end{pmatrix}
\]

where \( A_1 \) is a square matrix of order \( k \leq n \) if and only if the linear subspace spanned by the first \( k \) vectors of the basis \( a_1, \ldots, a_k \) is invariant under \( \varphi \);

\[
\begin{pmatrix}
A_1 & 0 \\
B & A_2
\end{pmatrix}
\]

where \( A_1 \) is a square matrix of order \( k \leq n \) if and only if the linear subspace spanned by the last \( n-k \) vectors of the basis \( a_{k+1}, \ldots, a_n \) is invariant under \( \varphi \);

\[
\begin{pmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_n
\end{pmatrix}
\]

where \( A_1 \) is a square matrix of order \( k \leq n \) if and only if the subspace spanned by the vectors \( a_1, \ldots, a_k \) and also the subspace spanned by the vectors \( a_{k+1}, \ldots, a_n \) are invariant under \( \varphi \).

*1503. Let the linear transformation \( \varphi \) of the \( n \)-dimensional space \( \mathbb{R}^n \) in the basis \( a_1, \ldots, a_n \) have a diagonal matrix with distinct elements on the diagonal. Find all the linear subspaces that are invariant under \( \varphi \) and determine their number.

1504. Find all the subspaces of three-dimensional space that are invariant under a linear transformation specified.
1516. Find all the subspaces of three-dimensional space and the invariant simultaneously under two linear transformations given by the matrices:

\[
\begin{pmatrix}
  5 & -1 & -1 \\
-1 & 5 & -1 \\
 1 & -4 & 5
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  -6 & 2 & 3 \\
  2 & -3 & 6 \\
  3 & 0 & 2
\end{pmatrix}
\]

1517. Prove that any two permutable linear transformations in a complex space have a common eigenvector.

1518. Prove that for any set (even infinite) of pairwise permutable linear transformations of the complex space \( \mathbb{C}^n \), there is an eigenvector for all transformations of the given set.

1519. Prove that all root vectors of distinct eigenvalues are linearly independent.

Find the eigenvalues and the root subspaces of the linear transformations given in some basis by the following matrices:

1520. \[
\begin{pmatrix}
  4 & 1 & 2 \\
 5 & 7 & 8 \\
 6 & 9 & 4
\end{pmatrix}
\]

1521. \[
\begin{pmatrix}
  2 & 6 & 15 \\
 1 & 5 & 4 \\
 12 & 6 & 1
\end{pmatrix}
\]

1522. \[
\begin{pmatrix}
  1 & 3 & 4 \\
 4 & -7 & 8 \\
 6 & 7 & 7
\end{pmatrix}
\]

1523. Prove that a linear transformation of complex space has a diagonal matrix in some basis if and only if all eigenvalues are distinct.

1524. Prove that a complex space consists solely of root vectors of a linear transformation if and only if all the root vectors of the transformation are equal.

1525. Suppose \( \mathbb{F} \) is an infinite-dimensional space of all functions, \( f(x) \) defined and having derivatives of any order over the entire number line, with ordinary addition of functions and ordinary multiplication of a function by a scalar; and suppose \( \phi \) is a transformation that carries any function into its derivative.

Find: (a) all eigenvalues and eigenvectors, (b) all root subspaces of the transformation \( \phi \).

1526. A space \( \mathbb{F} \) is said to be a cyclic space relative to a linear transformation \( \phi \) if \( \mathbb{F} \) has a cyclic basis, that is, a basis \( a_1, a_2, \ldots, a_n \) for which:

\[ \phi a_k = a_{k+1}, \quad (k = 1, 2, \ldots, n - 1) \]

1527. Prove that if \( \mathbb{F} \) is a cyclic space under \( \phi \) and \( a_1, a_2, \ldots, a_n \) is a cyclic basis, then

(a) the minimal polynomial \( g(\lambda) \) of the transformation \( \phi \) is of degree \( n \);

(b) the minimal polynomial of the whole space coincides with the minimal polynomial of the vector \( a_1 \);

(c) if \( \phi a_k = \alpha a_k + \ldots + \alpha a_1 + a_n \), then the minimal polynomial of the transformation \( \phi \) is defined by

\[ g(\lambda) = \lambda^n - \alpha n - 1 - \cdots - \alpha \]

1528. Prove that the degree of the minimal polynomial \( g(\lambda) \) of a linear transformation \( \phi \) of the space \( \mathbb{F} \), is \( n \) and the degree of a polynomial that is irreducible over the field over which the space \( \mathbb{F} \) is considered, that is, in the case of a complex space \( g(\lambda) = (\lambda - \alpha)^n \), then

(a) \( \mathbb{F} \) is not decomposable into a direct sum of two subspaces that are invariant under \( \phi \);

(b) \( \mathbb{F} \) is cyclic under \( \phi \).

What is the form of the matrix of the transformation \( \phi \) in a cyclic basis?

1529. Suppose the minimal polynomial of a linear transformation \( \phi \) of the space \( \mathbb{F} \), is of the form \( (\lambda - \alpha)^n \). Prove that there is a vector \( a \) such that the vectors \( \{\phi^k a : k = 0, 1, \ldots, n-1\} \) form a basis of the space. What is the form of the matrix of the transformation \( \phi \) in that basis?

1530. Prove that any subspace \( L \) of the complex space \( \mathbb{F} \), which subspace is invariant under the linear transformation \( \phi \), contains a straight line that is invariant under \( \phi \).
that is invariant under $\varphi$. Show, using examples, that this assertion does not hold for a subspace of even dimension.

Under "what conditions" does $L$ contain a straight line, all points of which remain fixed under the transformation $\varphi$?

1521. Prove that a complex space containing only one straight line is invariant under the linear transformation $\varphi$. It is decomposable into a direct sum of two nonzero subspaces that are invariant under $\varphi$.

1522. Prove that under a linear transformation $\varphi$ the complex space $\mathbb{R}^2$ decomposes into a direct sum of one or several invariant linear subspaces, each of which contains only one invariant straight line and, hence (by the preceding problem), cannot be decomposed further.

*1523. Let $\varphi$ be a linear transformation of the space $\mathbb{R}^2$ and let $g(\lambda)$ be the minimal polynomial of $\varphi$. Prove that

(a) if $g(\lambda) = h(\lambda) k(\lambda)$ and the polynomials $h(\lambda)$ and $k(\lambda)$ are coprime, then the space $\mathbb{R}^2$ is a direct sum of the subspaces $L_1$, which consists of all vectors $x$ such that $h(\varphi)x = 0$, and $L_2$, which consists of all vectors $x$ such that $k(\varphi)x = 0$;

(b) if $g(\lambda) = h_1(\lambda) h_2(\lambda) \ldots h_s(\lambda)$ and the polynomials $h_1(\lambda), h_2(\lambda), \ldots, h_s(\lambda)$ are pairwise coprime, then $\mathbb{R}^2$ is a direct sum of the subspaces $L_i$ ($i = 1, 2, \ldots, s$) where $L_i$ consists of all vectors $x$ such that $h_i(\varphi)x = 0$.

1524. A linear transformation $\varphi$ is given relative to the basis $e_1, e_2, e_3$ by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}.
$$

Find the minimal polynomial $g(\lambda)$ of the transformation and the decomposition of the space into a direct sum of subspaces, which decomposition corresponds to the decomposition of $g(\lambda)$ into coprime factors of the form $(\lambda - \alpha)^k$.

1525. Solve the same problem as that of 1524 if the linear transformation $\varphi$ in the basis $e_1, e_2, e_3$ is given by the matrix

$$
\begin{pmatrix}
4 & -2 & 2 \\
-5 & 7 & -5 \\
-6 & 6 & -4
\end{pmatrix}.
$$

1526. A linear transformation $\varphi$ of the Euclidean (unitary) space $\mathbb{R}_n$ is given by $\varphi x = (x, a) a$ for any $x$ in $\mathbb{R}_n$; $a$ is a given nonzero vector. Find the minimal polynomial $g(\lambda)$ of the transformation and the decomposition of the space into a direct sum, the decomposition corresponding to the decomposition of $g(\lambda)$ into coprime degrees of irreducible polynomials with real coefficients (or polynomials of the form $\lambda - \alpha$ in the case of a unitary space).

1527. Find the Jordan form of the matrix of a linear transformation $\varphi$ of the complex space $\mathbb{R}_n$ if $\varphi$ has, up to a numerical factor, only one eigenvector.

1528. Prove that the number of linearly independent eigenvectors of a linear transformation $\varphi$ that belong to one and the same eigenvalue $\lambda_0$ is equal to the number of blocks with diagonal elements $\lambda_0$ in the Jordan form of the matrix of $\varphi$.

*1529. Prove that the basis in which the matrix of the linear transformation $\varphi$ of the complex vector space $\mathbb{R}_n$ has Jordan form can be constructed in the following manner:

(A) If not all eigenvalues of $\varphi$ are equal and the characteristic polynomial is of the form

$$
f(\lambda) = (\lambda - \lambda_1)^{k_1} \ldots (\lambda - \lambda_s)^{k_s} \quad (\lambda_i \neq \lambda_j \text{ for } i \neq j),
$$

then we construct a basis of the subspace $P_i$ of all vectors $x$ such that $(\varphi - \lambda_i e)^{k_i}x = 0$ ($e$ is the identical transformation) ($i = 1, 2, \ldots, s$).

The space $\mathbb{R}_n$ will be a direct sum of the subspaces $P_i$. They are invariant under $\varphi$; $\varphi$ in $P_i$ has one eigenvalue $\lambda_i$, and $(\varphi - \lambda_i e)^{k_i}x = 0$ for any vector $x$ in $P_i$. In this construction, we can take the minimal polynomial $g(\lambda)$ instead of the characteristic polynomial $f(\lambda)$ but this may lower the exponents $k_i$.

(B) Let $\varphi$ in $\mathbb{R}_n$ have a unique eigenvalue $\lambda_0$ and let $k$ be the smallest positive integer such that $(\varphi - \lambda_0 e)^k = 0$. Set $\psi = \varphi - \lambda_0 e$. The height of the vector $x$ is the smallest $h$ such that $\psi^h x = 0$. We denote by $R_h$ the subspace of all vectors of height $\leq h$ ($0 \leq h \leq k$). $R_0$ contains only the zero vector; $R_k$ coincides with the entire space.

We construct the basis of $R_k$, complete it to the basis of $R_0$, which in turn is extended to $R_1$ and so on, until the basis of $R_k$ is completed (for brevity, we call these bases the initial bases). For each vector $f$ of height $k$ of the initial basis of $R_1$.
we construct a series of vectors $f, \psi f, \psi^2 f, \ldots, \psi^{k-1} f$ with initial vector $f$. We then take any basis of $R_h$ (say the initial basis) and vectors of height $k-1$ of all the constructed series. Taken together, these vectors are linearly independent. Now, using any vectors (say, from the initial basis of $R_h$) we complete them up to the basis of $R_h$. For each of the supplemental vectors $f$ (if they exist) we construct a new series: $f, \psi f, \psi^2 f, \ldots, \psi^{k-1} f$, and so forth.

Suppose that, at a certain stage, certain series have been constructed in which the vectors of height $h + 1$ together with any (say, the initial) basis of $R_h$ form a basis of $R_{h+1}$. The vectors of any (say, the initial) basis of $R_{h+1}$ together with vectors of height $h$ of the constructed series will be linearly independent. We complete them to the basis of $R_h$ using any vectors (say, from the initial basis of $R_h$). For each additionally taken vector $f$ (if such exist) we construct a new series $f, \psi f, \psi^2 f, \ldots, \psi^{k-1} f$. We proceed in this fashion until the vectors of all the constructed series form a basis of the entire space. Writing down the vectors one series after another so that in each series the vectors are taken in reverse order (the initial vector of a series is last in the given series), we get the desired basis in which the matrix of the transformation $\phi$ is of Jordan form.

(C) The basis whose construction is given in (A) and (B) is not defined uniquely. Prove the uniqueness (up to the order of arrangement of the Jordan submatrices) of the Jordan matrix $A_j$ which is similar to the given square matrix $A$ (and, hence, the uniqueness of the Jordan form of the matrix of the given linear transformation $\phi$). Namely, prove that the Jordan form $A_j$ of the matrix $A$ of order $n$ is determined as follows. Let $h$ be the highest order of the Jordan submatrices of the matrix $A_j$ with the number $\lambda_0$ on the diagonal, let $x_0$ be the number of such submatrices of order $h$ ($h = 1, 2, \ldots, k$), $B = A - \lambda_0 E$, and let $r_h$ be the rank of the matrix $B^h$ ($h = 0, 1, 2, \ldots, k, k + 1$). Then the numbers $x_h$ are given by the formulas

$$x_h = x_{h-1} - 2r_h + r_{h+1} \quad (h = 1, 2, \ldots, k).$$

Note. The formulas (a) offer a procedure for finding the Jordan form $A_j$ without applying to the theory of elementary divisors of $\lambda$ matrices.

For a linear transformation $\phi$ of space $R_n$ in the basis $e_1, e_2, \ldots, e_n$ is given by the matrix $A$. Find the basis $f_1, \ldots, f_n$ in which the matrix of the transformation has the Jordan form $A_j$ and find this Jordan form (the desired basis is not defined uniquely).

1530. $A = \begin{pmatrix} 3 & 2 & -3 \\ 4 & 10 & -12 \\ 3 & 6 & -7 \end{pmatrix}$

1531. $A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 3 & -3 \\ 3 & -2 & -2 \end{pmatrix}$

1532. $A = \begin{pmatrix} 0 & 3 & 3 \\ 2 & -14 & 10 \end{pmatrix}$

1533. $A = \begin{pmatrix} 6 & 6 & -15 \\ 1 & 5 & -5 \\ 4 & 4 & -2 \end{pmatrix}$

1534. $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

1535. $A = \begin{pmatrix} 8 & 17 & 14 & 6 & -9 & 5 \end{pmatrix}$

1536. $A = B^3$, where $B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$

is a Jordan submatrix of order $n$.

*1537. A linear transformation $\phi$ of the space $R_n$ is said to be an involutory transformation if $\phi^2 = e$, where $e$ is the identical transformation. Determine the geometrical meaning of an involutory transformation.

*1538. A linear transformation $\phi$ of the space $R_n$ is said to be idempotent if $\phi^2 = \phi$. Determine the geometrical meaning of an idempotent transformation.

1539. Give examples of a linear transformation $\phi$ of three-dimensional space for which

(a) the space is not a direct sum of the domain of values of $L_1$ and of the kernel of $L_2$ of the transformation $\phi$ (the definition is given in problem 1488);

(b) the space is a direct sum of the domain of values of $L_1$ and of the kernel of $L_2$ for $\phi$, but $\phi$ is not a projection on $L_1$ parallel to $L_2$. 

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Sec. 19. Linear Transformations of Euclidean and Unitary Vector Spaces

1540. Prove that the operation of passing from a linear transformation \( \varphi \) of unitary (or Euclidean) space to the conjugate transformation \( \varphi^* \) has the following properties:

(a) \( (\varphi^*)^* = \varphi \).
(b) \( (\varphi + \psi)^* = \varphi^* + \psi^* \).
(c) \( (\alpha \varphi)^* = \alpha \varphi^* \).
(d) If \( \varphi \) is nonsingular, then \( (\varphi^{-1})^* = (\varphi^*)^{-1} \).

1541. Suppose \( e_1 \) and \( e_2 \) constitute an orthonormal basis of a plane and the linear transformation \( \varphi \) in the basis \( f_1 = e_1, \ f_2 = e_1 + e_2 \) has the matrix \( \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \). Find the matrix of the conjugate transformation \( \varphi^* \) in the same basis \( f_1, f_2 \).

1542. A linear transformation \( \varphi \) of Euclidean space in a basis made up of the vectors \( f_1 = (1, 2, 1), \ f_2 = (1, 1, 2), \ f_3 = (1, 1, 0) \) is given by the matrix

\[
\begin{pmatrix}
1 & 1 & 3 \\
0 & 5 & -1 \\
2 & 7 & -3
\end{pmatrix}
\]

Find the matrix of the conjugate transformation \( \varphi^* \) in the same basis.

1543. Prove that the matrix of the linear transformation \( \varphi^* \) is obtained from the matrix of the linear transformation \( \varphi \) in the orthonormal basis \( e_1, e_2, e_3 \) of the vectors \( \alpha_1 = (0, 0, 1), \alpha_2 = (1, -1, 1), \alpha_3 = \) \( (1, 2, 1) \) into the vectors \( b_1 = (1, 2, 1), b_2 = (1, 1, 2), b_3 = (1, 1, 0) \), respectively, where the vectors \( b_1, b_2, b_3 \) are given in the basis \( e_1, e_2, e_3 \).

1544. Write down an orthonormal system of coordinates in the plane of the first and third quadrant, its origin the center of projection of the plane on the direction of the vector of the first and third quadrants.

1545. Let \( L_1 \) and \( L_2 \) be a decomposition of a Euclidean space into a direct sum of two subspaces; let \( \varphi \) be the projection of \( \mathbb{R}^n \) on \( L_1 \) and \( L_2 \) into the orthogonal complement of \( L_1 \) and \( L_2 \); \( \varphi^* \) is the conjugate transformation. Prove that \( H_1 = L_1^* = L_2^* \) and that \( \varphi^* \) of \( \mathbb{R}^n \) on \( L_1^* \) parallel to \( L_2^* \).

1546. Prove that if the subspace \( L \) of a unitary (or Euclidean) space is invariant under a linear transformation \( \varphi \) then the orthogonal complement \( L^* \) is invariant under the conjugate transformation \( \varphi^* \).

1547. Prove that the linear transformation \( \varphi \) of a unitary space \( \mathbb{R}^n \) has an invariant subspace of dimensions from zero to \( n \).

1548. Prove that for any linear transformation \( \varphi \) of a unitary space there is an orthonormal basis in which the matrix of that transformation has triangular form (Schur theorem).

1549. Write down the equation of a plane invariant under a linear transformation \( \varphi \) specified in some orthonormal basis by the matrix

\[
\begin{pmatrix}
4 & -23 & 17 \\
11 & -43 & 30 \\
15 & -54 & 37
\end{pmatrix}
\]

1550. Prove that if one and the same vector \( x \) is an eigenvector for the linear transformation \( \varphi \) with the eigenvalue \( \lambda \) and for the conjugate transformation \( \varphi^* \) with the eigenvalue \( \lambda_0 \), then \( \lambda \lambda_0 = 1 \).

1551. Prove that if a linear transformation \( \varphi \) of a unitary space \( \mathbb{R}^n \) has the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) then the eigenvalues of the conjugate transformation \( \varphi^* \) are the conjugate numbers \( \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n \).

1552. Prove that the corresponding coefficients of the minimal polynomials of conjugate linear transformations are conjugate to one another.

1553. Let a linear transformation \( \varphi \) of a Euclidean space in the basis \( e_1, e_2, e_3 \) and let the conjugate transformation \( \varphi^* \) in the reciprocal basis (see problem 1417) \( f_1, f_2, f_3 \) have the matrix \( B \). Prove that \( H = A^t \) in unitary space and \( H = A^t \) in Euclidean space.

1554. Let the scalar product on \( \mathbb{R}^n \) be given by the bilinear form \( \varphi \) with matrix \( B \). Then...
1561. Prove that if a linear transformation $\varphi$ of a unitary (or Euclidean) space preserves the lengths of all vectors, then it is unitary (or, respectively, orthogonal).

*1562. Given in a unitary (or Euclidean) space some transformation $\varphi$ under which each vector $\mathbf{x}$ is associated with a unique vector $\varphi\mathbf{x}$. Prove that if the transformation $\varphi$ preserves the scalar product, that is $\langle \varphi\mathbf{x}, \varphi\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for any vectors $\mathbf{x}, \mathbf{y}$ of the space, then $\varphi$ will be a linear and, hence unitary (or, respectively, orthogonal) transformation. Use examples to show that preservation of the scalar squares of all vectors is not sufficient for the linearity of $\varphi$.

1563. Suppose the scalar multiplication of the vectors of the space $R^n$ is given by the Gram matrix $U$ of the vectors of some basis. Find the condition that is necessary and sufficient for the linear transformation $\varphi$ specified in the same basis by the matrix $A$ to be

(a) an orthogonal transformation of Euclidean space;
(b) a unitary transformation of unitary space.

1564. Prove that if two vectors $\mathbf{x}$ and $\mathbf{y}$ of Euclidean (or unitary) space have the same length, then there is an orthogonal (or, respectively, unitary) transformation $\varphi$ that carries $\mathbf{x}$ into $\mathbf{y}$.

1565. Prove that if two pairs of vectors $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{y}_1, \mathbf{y}_2$ of Euclidean (or unitary) space have the properties $|\mathbf{z}_1| = |\mathbf{z}_2|$ and $|\mathbf{y}_1| = |\mathbf{y}_2|$ and the angle between $\mathbf{z}_1$ and $\mathbf{z}_2$ is equal to the angle between $\mathbf{y}_1$ and $\mathbf{y}_2$, then there is an orthogonal (respectively, unitary) transformation $\varphi$ such that $\varphi\mathbf{z}_1 = \mathbf{y}_1$, $\varphi\mathbf{z}_2 = \mathbf{y}_2$.

*1566. Given two sets of vectors $\mathbf{z}_1, \ldots, \mathbf{z}_k$ and $\mathbf{y}_1, \ldots, \mathbf{y}_k$ of Euclidean (or unitary) space. Prove the following: for an orthogonal (or, respectively, unitary) transformation $\varphi$ to exist such that $\varphi\mathbf{z}_i = \mathbf{y}_i$ ($i = 1, 2, \ldots, k$), it is necessary and sufficient for the Gram matrices of the sets of vectors to coincide:

$$(\mathbf{z}_i, \mathbf{z}_j)^T = (\mathbf{y}_i, \mathbf{y}_j)^T.$$
*1569. Prove that for a unitary transformation \( \varphi \) of unitary space

(a) the eigenvalues are equal to unity in modulus (and, hence, the eigenvalues of the unitary and, in particular, real orthogonal, matrix are equal to unity in modulus);

(b) the eigenvectors associated with two distinct eigenvalues are orthogonal;

(c) if in a certain basis the matrix \( A \) of the transformation \( \varphi \) is real and the eigenvector associated with the complex eigenvalue \( \alpha + \beta i \ (\beta \neq 0) \) is represented as \( x + yi \), where the vectors \( x \) and \( y \) have real coordinates (components), then \( x \) and \( y \) are orthogonal and have the same length; also note that \( qx = \alpha x - \beta y; \ qy = \beta x + \alpha y; \)

(d) an orthogonal transformation of Euclidean space always has a one-dimensional or two-dimensional invariant subspace.

*1570. Prove that

(a) for any unitary transformation \( \varphi \) of the unitary space \( R \), there is an orthonormal basis consisting of eigenvectors of the transformation \( \varphi \). In this basis, the matrix of \( \varphi \) is diagonal with diagonal elements equal to unity in absolute value.

What property of unitary matrices follows from this fact?

(b) for any orthogonal transformation \( \varphi \) of Euclidean space \( R \), there is an orthonormal basis in which the matrix of \( \varphi \) is of canonical form, where the principal diagonal has blocks of second order of the form

\[
\begin{pmatrix}
\cos \gamma & -\sin \gamma \\
\sin \gamma & \cos \gamma
\end{pmatrix} \quad (\gamma \neq k\pi)
\]

and blocks of first order of the form \((\pm 1)\).

Blocks of any of these types may be absent. All other elements are zero. What is the geometrical meaning of the transformation? What property of real orthogonal matrices follows from this fact?

Relative to an orthogonal transformation \( \varphi \) given in an orthonormal basis by the matrix \( A \), find the orthonormal basis in which the matrix \( \varphi \) of that transformation has the canonical form indicated in problem 1570. Find that canonical form. (The desired basis is not defined uniquely.)

\[
\begin{pmatrix}
\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{2}{3}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]
Sed и 1

ψ is an involutory transformation, that is, \( \psi^2 = e \) (\( e \) is the identical transformation), then it has the third property as well. Find all types of transformations possessing all these properties.

Find the orthonormal basis of eigenvectors and the matrix \( B \) in that basis for a linear transformation specified in some orthonormal basis by the matrix \( A \) (the desired basis is not defined uniquely):

\[
A = \begin{pmatrix}
11 & 2 & -3 \\
2 & 2 & 4 \\
-8 & 10 & 5
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
17 & -8 & 4 \\
-8 & 17 & -4 \\
4 & -4 & 11
\end{pmatrix}.
\]

For a given matrix \( A \) find the diagonal matrix \( B \) and the unitary matrix \( Q \) such that:

\[
B = Q^{-1}AQ.
\]

Prove that a linear combination of self-conjugate transformations with real coefficients (in particular, the sum of two self-conjugate transformations) is a self-conjugate transformation.

Prove that the product \( \psi \psi \) of two self-conjugate transformations \( \psi \) and \( \psi \) is self-conjugate if and only if \( \psi \) and \( \psi \) are permutable.

Prove that if \( \psi \) and \( \psi \) are self-conjugate transformations, then so also will the following be self-conjugate:

\[
\psi \psi + \psi \psi \text{ and } i \psi \psi - \psi \psi.
\]

Prove that the reflection of \( \psi \) of the Euclidean (or unitary) space \( R_n \) in the subspace \( L_1 \) parallel to the subspace \( L_2 \) is a self-conjugate linear transformation if and only if \( L_1 \) and \( L_2 \) are orthogonal.

Prove that the projection \( \psi \) of the Euclidean (or unitary) space \( R_n \) on the subspace \( L_1 \) parallel to the subspace \( L_2 \) is a self-conjugate linear transformation if and only if \( L_1 \) and \( L_2 \) are orthogonal.

Prove that if the linear transformation \( \psi \) of the Euclidean (or unitary) space \( R_n \) has any two of the following three properties:

(1) \( \psi \) is a self-conjugate transformation;
(2) \( \psi \) is a unitary (or, respectively, orthogonal) transformation;
(3) \( \psi \) is an involutory transformation, that is, \( \psi^2 = e \) (\( e \) is the identical transformation), then it has the third property as well. Find all types of transformations possessing all these properties.

Find the orthonormal basis of eigenvectors and the matrix \( B \) in that basis for a linear transformation specified in some orthonormal basis by the matrix \( A \) (the desired basis is not defined uniquely):

\[
A = \begin{pmatrix}
11 & 2 & -3 \\
2 & 2 & 4 \\
-8 & 10 & 5
\end{pmatrix};
\]

\[
A = \begin{pmatrix}
17 & -8 & 4 \\
-8 & 17 & -4 \\
4 & -4 & 11
\end{pmatrix}.
\]

For a given matrix \( A \) find the diagonal matrix \( B \) and the unitary matrix \( C \) such that \( B = C^{-1}AC \).

\[
A = \begin{pmatrix}
3 & 2i \\
2i & 1
\end{pmatrix};
\]

\[
A = \begin{pmatrix}
3 & 2i \\
2i & 7
\end{pmatrix}.
\]

Consider an \( n^2 \)-dimensional space of all complex square matrices of order \( n \) with the ordinary operations of adding matrices and multiplying a matrix by a scalar. Turn that space into a unitary space, assuming that the scalar product of two matrices \( \psi \) and \( \psi \) is given by the equation

\[
(\psi, \psi) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \overline{b_{ij}}.
\]

Prove that

(a) the premultiplication of all matrices by one and the same matrix \( C \) is a linear transformation;
(b) the unitary matrices have, as vectors of the indicated space, the length \( 1/n \);
(c) the premultiplication of all matrices by the conjugate transposes \( C \) and \( C' \) generates conjugate transformations;
(d) premultiplying by the unitary matrix \( C \) generates a unitary transformation;
(o) multiplying by a Hermitian matrix generates a self-conjugate transformation;
(f) multiplying by a skew-Hermitian matrix generates a skew-symmetric transformation.

1591. Suppose a scalar multiplication of vectors of the space $R_n$ is given by the Gram matrix $U$ of a certain basis. Find the necessary and sufficient condition for the linear transformation $\varphi$ specified in the same basis by the matrix $A$ to be self-conjugate in the case of (a) a Euclidean space, and (b) a unitary space.

*1592. Prove that two self-conjugate transformations $\varphi$ and $\psi$ of the unitary (or Euclidean) space $R_n$ has a common orthonormal basis of eigenvectors of both transformations if and only if the transformations commute. What property of quadratic forms and second-degree surfaces follow from this fact?

1593. Let $R$ be a Euclidean space of dimension $n^2$ whose vectors are all real matrices of order $n$ with ordinary addition of matrices and ordinary multiplication of a matrix by a scalar, and let the scalar product of the matrices $A = (a_{ij})$ and $B = (b_{ij})$ be given by $(AB) = \sum_{i,j=1}^n a_{ij}b_{ij}$.

Furthermore, let $P$ and $Q$ be real symmetric matrices of order $n^2$.

Prove that the linear transformations $\varphi X = PX$ and $\psi X = QX$ (X is any matrix of the space $R$) are permutable self-conjugate transformations of the space $R$, and determine the relationship between the common orthonormal basis of eigenvectors of the transformations $\varphi$ and $\psi$ and the orthonormal basis of eigenvectors of the matrices $P$ and $Q$.

1594. A self-conjugate linear transformation $\varphi$ of the unitary (or Euclidean) space $R_n$ is said to be positive definite if $(\varphi x, x) > 0$ and nonnegative if $(\varphi x, x) \geq 0$ for any vector $x \neq 0$ in $R_n$. Prove that the self-conjugate transformation $\varphi$ is positive definite (or nonnegative) if and only if all its eigenvalues are positive (or, respectively, nonnegative). Show that for any linear transformation $\varphi$ (and not only a self-conjugate transformation) $(\varphi x, x) > 0$ implies that all eigenvalues of $\varphi$ are positive (or, respectively, nonnegative). Give an example showing that the converse may not hold true for a nonself-conjugate linear transformation.

*1595. Prove that if $\varphi = \psi \chi$ or $\varphi = \chi \psi$, where $\varphi$ and $\psi$ are self-conjugate linear transformations with positive eigenvalues and $\chi$ is a unitary transformation, then $\varphi$ and $\psi$ are identical transformations (see problem 1276).

*1596. Prove that any nonsingular linear transformation of unitary (or Euclidean) space can be expressed in the form $\varphi = \psi_1 \chi_1$ and also in the form $\varphi = \chi_2 \psi_2$, where $\psi_1, \psi_2$ are self-conjugate transformations with positive eigenvalues and $\chi_1, \chi_2$ are unitary (or, respectively, orthogonal) transformations; note that both the indicated representations are unique.

1597. Explain why the equalities

\[
\begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 \\
0 & 2
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

are not in contradiction with the uniqueness of representation given in the preceding problem.

Represent the following matrices as products of a symmetric matrix with positive eigenvalues into an orthogonal matrix:

1598. \[
\begin{pmatrix}
2 & -1 \\
2 & 1
\end{pmatrix}
\]

1599. \[
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\]

1600. \[
\begin{pmatrix}
4 & -2 & 2 \\
4 & 1 & -1 \\
-2 & 4 & 2
\end{pmatrix}
\]

1601. Prove that the self-conjugate linear transformation $\varphi$ is positive definite if and only if the coefficients of its characteristic polynomial $\lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0$ are all nonzero and have alternating signs, and is not negative (that is, has nonnegative eigenvalues) if and only if the coefficients $c_0 = 1, c_1, c_2, \ldots, c_n$ are nonzero and have alternating signs and $c_{n+1}, \ldots, c_n$ are zero. Here, $k$ is any number from 0 to n.

*1602. Prove that if $\varphi$ and $\psi$ are self-conjugate transformations and $\varphi$ is positive definite, then the eigenvalues of the transformation $\varphi \psi$ are real.

*1603. Prove that if $\varphi$ and $\psi$ are self-conjugate transformations with nonnegative eigenvalues, one of which is nonsingular, then the eigenvalues of the transformation $\varphi \psi$ are real and nonnegative.
1604. Prove that the sum of two or several nonnegative self-conjugate transformations (see problem 1594) is again a nonnegative self-conjugate transformation.

1605. Prove that a nonnegative self-conjugate transformation of rank r is the sum of r nonnegative self-conjugate transformations of rank 1.

1606. Prove that a linear transformation q of the unitary space $\mathbb{K}^n$ having rank unity is a nonnegative self-conjugate transformation if and only if in any orthonormal basis its matrix can be represented as $X'X$, where $X$ is a row of $n$ numbers.

1607. Prove that if the matrices $X = (x_{ij})$ and $Y = (y_{ij})$ are Hermitian and nonnegative (that is, have nonnegative eigenvalues), then the matrix $T = (t_{ij})$, where $t_{ij} = x_{ij}y_{ij}$ for $i, j = 1, 2, \ldots, n$, is Hermitian and nonnegative (compare with problem 1220).

1608. A linear transformation $q$ of the Euclidean (or unitary) space $\mathbb{K}^n$ is said to be skew-symmetric if $q^\ast = -q$. Prove that a) if $q$ is a linear transformation of Euclidean space to be skew-symmetric, it is necessary and sufficient that its matrix $A$ in any orthonormal basis be skew-symmetric, that is, that $A' = -A$;
b) if a linear transformation $q$ of unitary space is to be skew-symmetric, it is necessary and sufficient for its matrix $A$ to be skew-Hermitian in any orthonormal basis, that is, $A' = -A$.

1609. Prove that the orthogonal complement $L^*$ of the linear subspace $L$ of a Euclidean (or unitary) space, which is invariant under the skew-symmetric transformation $q$, is also invariant under $q$.

1610. Prove that for the skew-symmetric transformation $g$ of a unitary space,$g$ the eigenvalues are pure imaginaries (and, hence, the eigenvalues of a complex skew-Hermitian matrix, in particular all skew-symmetric matrices, are pure imaginaries).

1611. Prove that a skew-symmetric transformation of a Euclidean space always has an one-dimensional or two-dimensional invariant subspace.

(a) For any skew-symmetric transformation $q$ of the unitary space $\mathbb{K}^n$, there exists an orthonormal basis consisting of eigenvectors of the transformation $q$. In this basis, the matrix of $q$ is diagonal with purely imaginary elements on the diagonal (some of these elements may be zero). What property of complex skew-Hermitian matrices does this imply?

(b) For any skew-symmetric transformation $q$ of the Euclidean space $\mathbb{K}^n$, there exists an orthonormal basis, in which the matrix has the following canonical form on the main diagonal are second order blocks of the form

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

where $| \lambda | \neq 0$, and first order zero blocks (one of the two types of block may be absent). What is the geometrical meaning of the transformation? What property of real skew-symmetric matrices does this imply?

1612. Prove that if $q$ is a self-conjugate transformation of a unitary space, then the transformation $q^\ast = -q$ is skew-symmetric. Conversely, if $q$ is skew-symmetric, then $q^\ast$ is a self-conjugate transformation.

1613. Prove that if $q$ is a self-conjugate transformation of a unitary space, then the transformation $q^\ast = (q - q)^\ast (q + q)$, where $q$ is the identical transformation, exists and is unitary.

1614. Prove that the skew-symmetric and unitary transformations of a Euclidean space (and, respectively, the skew-symmetric and orthogonal transformations of a Euclidean space) are related as follows: If in the equation

\[
\psi = (q - q)(q + q)^\ast
\]

(1)

(where $q$ is the identical transformation) $q$ is a skew-symmetric transformation, then $q$ is a unitary transformation in which there is no eigenvalue of 1. Conversely, if in that same equation (1), $q$ is a unitary transformation devoid of an eigenvalue of 1, then $q$ is a skew-symmetric transformation.
formation. The equation (1) defines a reciprocal one-to-one mapping of all skew-symmetric transformations into all unitary transformations devoid of the eigenvalue \(-1\). There is a similar relationship between skew-symmetric and orthogonal transformations of Euclidean space. What properties of matrices does this imply?

1615. Show that the equation (1) of the preceding problem defines a one-to-one correspondence, firstly, between all nonsingular skew-symmetric transformations and all unitary (or, respectively, orthogonal) transformations devoid of eigenvalues \(\pm 1\), and, secondly, between all singular skew-symmetric transformations and all unitary (orthogonal) transformations having as eigenvalues \(+1\) and not having as eigenvalues \(-1\).

1616. Prove that if \(\varphi\) is a skew-symmetric transformation of a unitary (or Euclidean) space, then the transformation \(e^\varphi\) is unitary (respectively, orthogonal). What property of matrices does this imply?

*1617. Prove that the function \(e^\varphi\) generates a one-to-one mapping of all self-conjugate transformations of a unitary (or Euclidean) space onto all positive definite transformations (that is, such that are self-conjugate with positive eigenvalues).

1618. A linear transformation \(\varphi\) of a unitary (or Euclidean) space is said to be normal if it is permutable with its conjugate transformation \(\varphi^*\). Verify that self-conjugate, skew-symmetric, and unitary (or orthogonal) transformations are normal.

1619. Prove that a normal transformation of a unitary (or Euclidean) space is self-conjugate if and only if all its eigenvalues (respectively, all the roots of its characteristic equation) are real.

1620. Prove that the normal transformation of a unitary (or Euclidean) space is unitary (or, respectively, orthogonal) if and only if all its eigenvalues (or, respectively, all the roots of its characteristic equation) are equal to unity in absolute value.

1621. Prove that the normal transformation of a unitary (or Euclidean) space is skew-symmetric if and only if all its eigenvalues (or, respectively, all the roots of its characteristic equation) are pure imaginaries.

1622. Prove that a linear transformation \(\varphi\) of a unitary space is normal if and only if \(\varphi = \psi\chi\), where \(\varphi\) is a self-conjugate transformation and \(\chi\) is an unitary transformation and these transformations commute.

1623. Prove that
(a) every linear transformation \(\varphi\) can be uniquely represented as \(\varphi = \varphi_1 + \varphi_2\), where \(\varphi_1\) is a self-conjugate transformation and \(\varphi_2\) is a skew-symmetric transformation.
(b) for a transformation \(\varphi\) to be normal, it is necessary and sufficient that the transformations \(\varphi_1\) and \(\varphi_2\) in the preceding representation be permutable.

1624. Prove that
(a) every linear transformation \(\varphi\) of a unitary space can be uniquely represented in the form \(\varphi = \varphi_1 + \varphi_2\), where \(\varphi_1\) and \(\varphi_2\) are self-conjugate transformations.
(b) for the transformation \(\varphi\) to be normal, it is necessary and sufficient for the transformations \(\varphi_1\) and \(\varphi_2\) in the preceding representation to be permutable.

*1625. Prove that for any (finite or infinite) collection of pairwise permutable normal transformations of the unitary space \(R^n\) there is an orthonormal basis whose vectors are eigenvectors for all transformations of the given collection.

1626. Prove that it is possible, relative to any normal transformation \(\varphi\) of the unitary space \(R^n\), to take the \(k\)th root for any natural number \(k\) in the domain of normal transformations. Find the number of distinct normal transformations \(\varphi\) such that \(\varphi^k = \varphi\).

*1627. Prove that if \(x\) is an eigenvector of the normal transformation \(\varphi\) of a unitary (or Euclidean) space, which eigenvector is associated with the eigenvalue \(\lambda\), then \(x\) is an eigenvector of the conjugate transformation \(\varphi^*\), which eigenvector is associated with the conjugate (respectively, very same) eigenvalue \(\overline{\lambda}\).

*1628. Prove that the eigenvectors of a normal transformation that are associated with two distinct eigenvalues are orthogonal.

*1629. Let \(e\) be the eigenvector of a normal transformation \(\varphi\). Prove that the subspace \(L\) consisting of all vectors of the space that are orthogonal to \(e\) are invariant under \(\varphi^*\).

*1630. Prove that to achieve normality in the linear transformation \(\varphi\) of a unitary space it is necessary and sufficient for each eigenvector of \(\varphi^*\) to be an eigenvector of \(\varphi\) as well.
1631. Prove that any subspace $L$ of the unitary space $R_n$, which is invariant under the normal transformation $\varphi$, has an orthonormal basis consisting of the eigenvectors of the transformation $\varphi$.

1632. A linear transformation $\varphi$ of the unitary (or Euclidean) space $R_n$ is said to have a normal property if the orthogonal complement $L^*$ of every subspace $L$, which is invariant under $\varphi$, is itself invariant under $\varphi$. Prove the following assertion: for a linear transformation $\varphi$ of a unitary (or Euclidean) space to be normal, it is necessary and sufficient that $\varphi$ have the normal property.

1633. Prove that for normality of a linear transformation $\varphi$ of a unitary (or Euclidean) space it is necessary and sufficient that every subspace that is invariant under $\varphi$ be invariant under $\varphi^*$ as well.

Supplement

Sec. 20. Groups

1634. Determine whether each of the following sets forms a group under the indicated operation on the elements of the set:

1. the integers under addition;
2. even numbers under addition;
3. the integers that are multiples of a given natural number $n$ under addition;
4. the powers of a given real number $a, a \neq 0, \pm 1$ with integral exponents under multiplication;
5. nonnegative integers under addition;
6. odd integers under addition;
7. the integers under subtraction,
8. the rational numbers under addition;
9. the rational numbers under multiplication;
10. the nonzero rationals under multiplication;
11. the positive rationals under multiplication;
12. the positive rationals under division;
13. the dyadic-rational numbers, that is, rational numbers whose denominators are powers of the number 2 with integral nonnegative exponents, under addition;
14. all rational numbers whose denominators are equal to products of primes taken from the given set $M$ (finite or infinite) with integral nonnegative exponents (only a finite number of which can be nonzero), under addition;
15. the $n$th roots of unity (both real and complex roots), under multiplication;
16. the roots of unity of all positive integral powers, under multiplication;
17. $n$th-order matrices with real elements, under multiplication;
(18) nonsingular \( n \times n \) matrices with real elements, under multiplication;
(19) \( n \times n \) matrices with integral elements, under multiplication;
(20) \( n \times n \) matrices with integral elements and with determinant unity, under multiplication;
(21) \( n \times n \) matrices with integral elements and determinant equal to \( \pm 1 \), under multiplication;
(22) \( n \times n \) matrices with real elements, under addition;
(23) substitutions of the numbers 1, 2, ..., \( n \), under multiplication;
(24) even substitutions of the numbers 1, 2, ..., \( n \), under multiplication;
(25) odd substitutions of the numbers 1, 2, ..., \( n \), under multiplication;
(26) one-to-one mappings of the set \( N = \{1, 2, 3, \ldots \} \) of positive integers onto itself, each mapping shifting only a finite number of numbers if for the product of mappings \( s \) and \( t \) we assume the mapping \( st \), which is obtained when the mappings \( s \) and \( t \) are carried out in succession;
(27) transformations of the set \( M \), that is, the one-to-one mappings of that set onto itself, if for the product of the transformations \( s \) and \( t \) we take the transformation \( st \), which is obtained when the transformations \( s \) and \( t \) are carried out in succession;
(28) vectors of the \( n \)-dimensional linear space \( \mathbb{R}^n \), under addition;
(29) parallel translations of three-dimensional space \( \mathbb{R}^3 \) if for the product of the translations \( s \) and \( t \) we take their translations in succession;
(30) rotations of three-dimensional space \( \mathbb{R}^3 \) about a given point \( O \) if for the product of the rotations \( s \) and \( t \) we take their successive execution;
(31) all motions of three-dimensional space \( \mathbb{R}^3 \) if for the product of the motions \( s \) and \( t \) we take the motion \( st \) that results in a successive execution of the motions \( s \) and \( t \);
(32) the positive real numbers if the operation is defined
\[ a \cdot b = ab; \]
(33) the positive real numbers if the operation is defined
\[ a \cdot b = e^x b; \]
(34) real polynomials of degree \( \leq n \) (including zero) in the unknown \( x \), under addition;
(35) real polynomials of degree \( \leq n \) in the variable \( x \), under addition;
(36) real polynomials of any degrees (including zero) in the unknown \( x \), under addition.

1635. Prove that a finite set \( G \) in which an associative algebraic operation is defined and each of the equalities
\[ ax = b \quad \text{and} \quad yx = b \]
for arbitrary \( a, b, x, y \) in \( G \) has at most one solution is a group.
1636. Prove that if \( a^2 = e \) for any element \( a \) of the group \( G \), then it is an Abelian group.
1637. Prove that the group of \( n \)-th roots of unity is a unique multiplicative group of the \( n \)-th order with numerical elements.
1638. Find all groups (up to within an isomorphism) of order (a) 3, (b) 4, (c) 6. Write a multiplication table for these groups and represent them in the form of substitution groups.
1639. Show that the rotations of each of five regular polyhedrons about the centre, which rotations bring coincidence the polyhedron with itself, form a group of the multiplication (composition) of two rotations we take the rotations in succession. Find the orders of the group.
1640. Prove that the groups (1) to (4) in problem 1637 are isomorphic.
1641. Prove that
(a) all infinite cyclic groups are isomorphic among themselves;
(b) all finite cyclic groups of a given order \( n \) are isomorphic among themselves.
1642. Prove that
(a) the group of positive real numbers under multiplication is isomorphic with the group of all real numbers under addition;
(b) the group of positive rational numbers under multiplication is not isomorphic with the group of all rational numbers under addition.
1643. Prove that
(a) any finite group of order \( n \) is isomorphic with a certain substitution group of \( n \) elements;
(b) any group is isomorphic with the group of certain one-to-one mappings of the set of elements of the group onto itself
1644. Prove that for any elements \( a, b, c \) of a group \( G \)
(a) the elements \( ab \) and \( ba \) are of the same order;
(b) the elements \( abc, bca \) and \( cab \) are of the same order.

1645. Prove that if \( e \) is unity and \( a \) is an element of order \( n \) of a group \( G \), then \( a^k = e \) if and only if \( k \) is divisible by \( n \).

1646. Let \( G = \{a\} \) be a cyclic group of order \( n \) and \( b = a^k \). Prove that
(a) the element \( b \) is a generator of the group \( G \) if and only if the numbers \( n \) and \( k \) are coprime;
(b) the order of the element \( b \) is equal to \( \frac{n}{d} \), where \( d \) is the greatest common divisor of \( n \) and \( k \);
(c) if \( n \) and \( k \) are coprime, then there is a root \( \sqrt[k]{a} \) in \( G \), that is, \( a \) is the \( k \)th power of some element of \( G \) and conversely;
(d) all elements are squares in a group of odd order.

1648. Prove the following assertions:
(a) if the elements \( a \) and \( b \) of a group \( G \) are permutable, that is, \( ab = ba \) (1)
and have finite coprime orders \( r \) and \( s \), then their product \( ab \) is of order \( rs \);
(b) if the elements \( a \) and \( b \) of a group \( G \) are permutable, have finite orders \( r \) and \( s \) and the intersection of their cyclic subgroups contain the unit element \( e \) alone, that is \( \{a\} \cap \{b\} = \{e\} \), (2) then the order of the product \( ab \) is equal to the least common multiple of \( r \) and \( s \). Use examples to show that for this assertion to hold, the conditions (1) and (2) separately are not sufficient and the condition (1) is not a consequence of condition (2) even for coprime orders of the elements \( a \) and \( b \).
(c) if the orders \( r \) and \( s \) of the elements \( a \) and \( b \) are coprime, then condition (2) holds;
(d) use an example to show that without condition (2) the order of the product \( ab \) cannot be determined uniquely by the orders of the factors \( a \) and \( b \).

1649. Which groups of problem 1634 are subgroups of the other groups?
*1657. Find all subgroups of the primary cyclic group, that is, of the cyclic group $G = \{a\}$ of order $p^n$, where $p$ is prime.

*1658. Prove the following statements:
(a) a symmetric group $S_n$ for $n > 1$ is generated by the set of all transpositions $(i, j)$;
(b) the symmetric group $S_n$ for $n > 1$ is generated by the transpositions: $(1, 2), (1, 3), \ldots, (1, n)$;
(c) the alternating group $A_n$ for $n > 2$ is generated by the set of all triple cycles $(i j k)$;
(d) the alternating group $A_n$ for $n > 2$ is generated by the triple cycles: $(123), (124), \ldots, (12n)$.

1659. Find the cosets:
(a) of the additive group of integers with respect to the subgroup of numbers that are multiples of a given natural number $n$;
(b) of the additive group of real numbers with respect to the subgroup of integers;
(c) of the additive group of complex numbers with respect to the subgroup of Gaussian integers, that is, the numbers $a + bt$, where $a$ and $b$ are integers;
(d) of the additive group of vectors in the plane (vectors issuing from the coordinate origin) with respect to the subgroup of vectors lying on the axis of abscissas Ox;
(e) of the multiplicative group of complex numbers different from zero with respect to the subgroup of numbers equal to unity in absolute value;
(f) of the multiplicative group of complex numbers different from zero with respect to the subgroup of positive real numbers;
(g) of the multiplicative group of complex numbers different from zero with respect to the subgroup of real numbers;
(h) of the symmetric group $S_n$ with respect to the subgroup of substitutions that leave the number $n$ fixed.

*1660. Prove that
(a) the subgroup $H$ of order $k$ of the finite group $G$ of order $2k$ contains the squares of all elements of $G$;
(b) the subgroup $H$ of index 2 of any group $G$ contains the squares of all elements of $G$.

*1661. Prove that for $n > 1$ the alternating group $A_n$ is the sole subgroup of index two (that is, such that contains half of all the elements) in the symmetric group $S_n$.

1662. Prove that:
(a) the group of a tetrahedron is isomorphic to the group of even substitutions of four elements;
(b) the groups of a cube and an octahedron are isomorphic to the group of all substitutions of four elements;
(c) the groups of the cube and the octahedron are isomorphic to the group of even substitutions of five elements. For the definition of polyhedron groups see problem 1639.

1663. Prove that any subgroup of index two is a normal divisor.

1664. Prove that the set $X$ of all elements of a group is each of which is permutable with all elements of that group is a normal divisor (the centre of the group $G$).

1665. The element $aba^{-1}b^{-1}$ is termed a commutator of the elements $a$ and $b$ of a group $G$. Prove that all commutators and their products (with any finite number of factors) form a normal divisor $K$ of the group $G$ (the commutant of the given group).

1666. Prove that in the group of all motions of three-dimensional space the element $x^{-1}ax$, that is, conjugate to the rotation $a$ about a point $P$, is a rotation about a point $Q$ into which $P$ passes under the motion $x$.

1667. Prove that the substitution $x^{-1}ax$, which is conjugate to the substitution $a$ in the substitution group, is obtained by applying a transforming substitution $x$ to all numbers in the expansion of the substitution $a$ into independent cycles.

*1668. Prove that
(a) the four-group $V$ (problem 1638) is a normal divisor of the symmetric group $S_4$;
(b) the factor group $S_4/V$ is isomorphic to the symmetric group $S_3$.

*1669. Using problem 1667, find the number of substitutions of the symmetric group $S_n$ that are permutable with the given substitution $a$.

1670. Prove that if the intersection of two normal subgroups $H_1$ and $H_2$ of a group $G$ contains only the unit $e$, then any element $h_1 \in H_1$ is permutable with any element $h_2 \in H_2$.

1671. Prove that
(a) the elements of a group $G$ that are permutable with the
given element $a$ form a subgroup $N(a)$ of the group $G$ (the
normalizer $N$ in $G$) that contains the cyclic subgroup $(a)$
as a normal divisor;
(b) the number of elements of the group $G$ that are conju-
gate to $a$ is equal to the index of the normalizer $N(a)$ in $G$.
1672. Prove that
(a) the elements of the group $G$ that are permutable with
the given subgroup $H$ (but not necessarily with the ele-
ments of $H$) form a subgroup $N(H)$ of the group $G$ (the
normalizer of the subgroup $H$ in $G$) that contains the
subgroup $H$ as a normal divisor;
(b) the number of subgroups of the group $G$ that are con-
jugate to $H$ is equal to the index of the normalizer $N(H)$
in $G$.
1673. Prove that
(a) the number of elements in the group $G$ that are conju-
gate to the given element divides the order of $G$;
(b) the number of subgroups of the group $G$ that are conju-
gate to the given subgroup divides the order of the group $G$.
1674. Use the problems 1669 and 1671 to find the number
of conjugations of the symmetric group $S_n$ that are conju-
gate to the given substitution $a$.
1675. Prove that the centre of a group $G$ of order $p^n$,
where $p$ is prime, contains more than one element.
1676. Prove that any normal divisor $H$ of the alternating
group $A_n$ of degree $n > 5$, which divisor contains at least
one triple cycle, coincides with $A_n$.
1677. (a) Find all classes of conjugate elements of the
icosahedron group (problem 1633); (b) prove that the ico-
sahedron group is simple (that is, that it does not contain
any normal divisor different from the group itself and from
the unit subgroup).
1678. Prove that an alternating group of the fifth degree
is simple.
1679. Prove that the group $G'$ is a homomorphic image
of a finite cyclic group $G$ if and only if $G'$ is also cyclic and
its order divides the order of the group $G$.
1680. Prove that if a group $G$ is homomorphically mapped
onto the group $G'$, an element $a$ in $G$ being mapped onto $a'$
in $G'$, then
(a) the order of $a$ is divisible by the order of $a'$;
b) the order of $G$ is divisible by the order of $G'$.
1681. Find all the homomorphic mappings:
(a) of a cyclic group $(a)$ of order $n$ into itself;
(b) of a cyclic group $(a)$ of order 6 into a cyclic group
$(b)$ of order 18;
(c) of a cyclic group $(a)$ of order 18 into a cyclic group
$(b)$ of order 6;
(d) of a cyclic group $(a)$ of order 12 into a cyclic group
$(b)$ of order 15;
(e) of a cyclic group $(a)$ of order 6 into a cyclic group
$(b)$ of order 25.
1682. Prove that the additive group of rational numbers
cannot be homomorphically mapped onto the additive
group of integers.
1683. An isomorphic mapping of a group $G$ onto itself
is termed an automorphism, and a homomorphic mapping
into itself is termed an endomorphism of that group. An auto-
morphism $\phi$ is said to be internal if there is an element
in $G$ such that $\phi a = x a x^{-1}$ for any $a$ in $G$, and it is said
to be external otherwise. All automorphisms of $G$ them-
selves form a group if a product of automorphisms is defined
as the automorphisms taken in succession: $\phi \psi(a) = \phi(\psi(a))$. All
endomorphisms of an Abelian group $G$ form a ring if
the addition of endomorphisms is defined by the equality
$\psi + \phi = \psi(a) + \phi(a)$, and multiplication is defined in the
same way as for automorphisms. Find the group of auto-
morphisms of a cyclic group $(a)$ of order $(a) 5$ and $(b) 6$.
(c) Prove that the symmetric group $S_3$ has six internal
automorphisms and not a single external one, and the group
of automorphisms is isomorphic to $S_3$.
(d) The four-group $V$ (problem 1638) has one internal
automorphism (identical) and five external ones, and the
group of automorphisms is isomorphic to $S_3$.
Find the ring of endomorphisms of the cyclic group $(a)$
of order $(a) 5$, $(f) 6$, $(g) n$.
1684. Prove that the factor group of a symmetric group
$S_n$ with respect to an alternating group $A_n$ is isomorphic
to the factor group of the additive group of integers with
respect to the subgroup of even numbers.
1685. Find the factor groups:
(a) of the additive group of integers with respect to the
subgroup of multiples of the given natural number $n$;
b) of the additive group of integers that are multiples
of 3 with respect to the subgroup of multiples of 15;
(c) of the additive group of integers that are multiples of 4 with respect to the subgroup of multiples of 24;
(d) of the multiplicative group of real numbers different from zero with respect to the subgroup of positive numbers.

1686. Let \( G_n \) be the additive group of vectors of an \( n \)-dimensional linear space and let \( H_n \) be the subgroup of vectors of a \( k \)-dimensional subspace, \( 0 \leq k \leq n \). Prove that the factor group \( G_n/H_n \) is isomorphic to \( G_{n-k} \).

1687. Let \( G \) be the multiplicative group of all complex numbers different from zero and let \( H \) be the set of all numbers in \( G \) lying on the real and imaginary axes.
(a) Prove that \( H \) is a subgroup of the group \( G \).
(b) Find the cosets of the group \( G \) with respect to the subgroup \( H \).
(c) Prove that the factor group \( G/H \) is isomorphic to the multiplicative group \( U \) of all complex numbers equal to unity in absolute value, and \( U_n \) the multiplicative group of the \( n \)th roots of unity. Prove that

1688. Let \( G \) be the multiplicative group of complex numbers different from zero and let \( H \) be the set of numbers in \( G \) that lie on \( n \) rays issuing from zero at equal angles. (a) \( K \) be the additive group of all real numbers, \( Z \) be the additive group of integers, \( D \) the multiplicative group of positive numbers, \( U \) the multiplicative group of complex numbers equal to unity in absolute value, and \( U_n \) the multiplicative group of the \( n \)th roots of unity. Prove that

1689. For multiplicative groups of nonsingular square matrices of order \( n \), prove the following statements:
(a) the factor group of the group of real matrices with respect to the subgroup of matrices with determinant unity is isomorphic to the multiplicative group of real numbers different from zero;
(b) the factor group of the group of real matrices with respect to the subgroup of matrices with determinant equal to \( \pm 1 \) is isomorphic to the multiplicative group of positive numbers;
(c) the factor group of the group of real matrices with respect to the subgroup of matrices with positive determinants is a cyclic group of the second order;
(d) the factor group of the group of complex matrices with respect to the subgroup of matrices with determinant equal to unity in absolute value is isomorphic to the multiplicative group of positive numbers;
(e) the factor group of the group of complex matrices with respect to the subgroup of matrices with positive determinants is isomorphic to the multiplicative group of complex numbers equal to unity in absolute value.

1690. Let \( G \) be the group of all motions of three-dimensional space, \( H \) the subgroup of parallel translations, and \( U \) the subgroup of rotations about a given point \( O \). Prove that

1691. Prove that the normal divisor \( H \) of the group \( G \) is isomorphic to \( K \).

1692. Prove that the factor group \( G/H \) is commutative if and only if \( H \) contains the commutant \( K \) of the group \( G \) (problem 1665).

*1693. Prove that the factor group of a noncommutative group \( G \) with respect to its centre \( Z \) (problem 1664) cannot be cyclic.

*1694. Prove that if the order of a finite group \( G \) is divisible by the prime number \( p \), then \( G \) contains an element of order \( p \) (Cauchy's theorem).

*1695. Let \( p \) be a prime. The group \( G \) is called a \( p \)-group (in the commutative case, a primary group) if the orders of all its elements are finite and are equal to certain powers of the number \( p \). Prove that the finite group \( G \) is a \( p \)-group if and only if its order is equal to the power of the number \( p \).

1696. Prove that
(a) the additive group of vectors of \( n \)-dimensional linear space is a direct sum of \( n \) subgroups of the vectors of \( n \)-dimensional subspaces spanned by the vectors of any basis of space;
the additive group of complex numbers is a direct sum of the subgroups of real numbers and pure imaginaries;
(c) the multiplicative group of real numbers is a direct product of the subgroup of positive numbers and the subgroup of numbers \( \pm 1 \);
(d) the multiplicative group of complex numbers is the direct product of the subgroups of positive numbers and of numbers equal to unity in absolute value.

1697. Prove that if \( G = A + B_1 = A + B_2 \) are direct decompositions of the Abelian group \( G \) and if \( B_1 \) contains \( B_2 \), then \( B_1 = B_2 \).

1698. Prove that the subgroup \( H \) of the Abelian group \( G \) is a summand in the direct decomposition \( G = H + K \) if and only if there is a homomorphic mapping of \( G \) onto \( H \) that holds all elements of \( H \) fixed.

1699. Prove that if \( G = A + B \) is a direct decomposition of the group \( G \), then the factor group \( G/A \) is isomorphic to \( B \).

1700. Let \( G = A_1 + A_2 + \ldots + A_s \) be a decomposition of the Abelian group \( G \) into a direct sum of the subgroups and let

\[ x = a_1 + a_2 + \ldots + a_s, \quad a_i \in A_i, \quad i = 1, 2, \ldots, s \]

be the appropriate decomposition of the element \( x \) into a sum of components. Prove that

(a) the group \( G \) is of finite order \( n \) if and only if each subgroup \( A_i \) has finite order \( n_i \), \( i = 1, 2, \ldots, s \), and \( n = n_1 n_2 \ldots n_s \);
(b) the element \( x \) is of finite order \( p \) if and only if each of its components \( a_i \) is of finite order \( p_i \), \( i = 1, 2, \ldots, s \), and \( p \) is equal to the lowest common multiple of the numbers \( p_1 p_2 \ldots p_s \);
(c) the group \( G \) is a finite cyclic group if and only if all the direct summands \( A_i \) are finite cyclic groups and their orders are relatively prime in pairs.

1701. Decompose into a direct sum of primary cyclic subgroups the cyclic group \( \langle a \rangle \) of order \( (a) \ 6 \), \( (b) \ 12 \), \( (c) \ 60 \)

1702. Prove the nondecomposibility into a direct sum of two nonzero subgroups:
(a) of the additive group of integers;
(b) of the additive group of rational numbers;
(c) of the primary cyclic group.

1703. Let \( G \) be a nonzero finite Abelian group (with additive notation for the group operation). Prove the following statements:

(a) if the orders of all elements of \( G \) divide the product \( pq \) of coprime numbers \( p \) and \( q \), then \( G \) can be decomposed into a direct sum of subgroups \( A \) and \( B \), where the orders of all elements of \( A \) divide \( p \), and those of \( B \) divide \( q \); note that one of the subgroups, \( A \) or \( B \), may prove to be zero;
(b) for the group \( G \) we have the decomposition \( G = A_1 + A_2 + \ldots + A_s \) into a direct sum of (nonzero) primary subgroups that refer appropriately to distinct primes \( p_1, p_2, \ldots, p_s \). These subgroups \( A_i \) are called primary components of the group \( G \);
(c) the primary component \( A_i \) which refers to the prime number \( p_i \) consists of all elements of the group \( G \) whose orders are equal to the powers of the number \( p_i \), which fact uniquely determines the decomposition of the group \( G \) into primary components;
(d) the decomposition, into primary components, of the nonzero subgroup \( H \) of the group \( G \) is of the form \( H = B_1 + B_2 + \ldots + B_s \), where \( B_i = H \cap A_i, \ i = 1, 2, \ldots, s \); the zero subgroups \( B_i \) in the decomposition of \( H \) are omitted.

1704. Denote by \( G (n_1, n_2, \ldots, n_s) \) the direct sum of the cyclic groups of orders \( n_1, n_2, \ldots, n_s \) respectively. From the theory of finite Abelian groups we know that each such group can be uniquely (up to an isomorphism) represented in the form \( G (n_1, n_2, \ldots, n_s) \), where the numbers \( n_i \) are equal to the powers of prime numbers (not necessarily distinct). Apply this notation to find all Abelian groups of the following orders: \( (a) \ 3, (b) \ 4, (c) \ 6, (d) \ 8, (e) \ 9, (f) \ 12, (g) \ 16, (h) \ 24, (i) \ 30, (j) \ 36, (k) \ 48, (l) \ 60, (m) \ 63, (n) \ 72, (o) \ 100 \).

1705. Decompose into a direct sum of primary cyclic and infinite cyclic subgroups, the factor group \( G/H \), where \( G \) is a free Abelian group with basis \( x_1, x_2, x_3 \) and \( H \) is a subgroup with the generators:

(a) \[ y_1 = 7x_1 + 2x_2 + 3x_3, \quad y_1 = 4x_1 \quad 5x_2 \quad 3x_3; \]
(b) \[ y_2 = 21x_1 + 8x_2 + 9x_3, \quad y_2 = 5x_1 + 6x_2 \quad 5x_3; \]
(c) \[ y_3 = 5x_1 - 4x_2 + 3x_3, \quad y_3 = 8x_1 + 7x_2 - 9x_3; \]
Determine which of the following sets are rings (and fields) and which are fields with respect to the indicated operations. (If the operations are not indicated, they assume addition and multiplication of the numbers.)

1709. The integers.
1710. The even numbers.
1711. Integers that are multiples of \( n \) (consider, for instance, the case \( n = 0 \)).
1712. The rational numbers.
1713. The real numbers.
1714. The complex numbers.
1715. Numbers of the form \( a + b \sqrt{2} \), where \( a \) and \( b \) are integers.
1716. Numbers of the form \( a + b \sqrt{3} \), where \( a \) and \( b \) are rational.
1717. Complex numbers of the form \( a + bi \) where \( a \) and \( b \) are integers.
1718. Complex numbers of the form \( a + bi \) where \( a \) and \( b \) are rational.
1719. Matrices of order \( n \) with integral elements, under addition and multiplication of matrices.
1720. Matrices of order \( n \) with real elements, under addition and multiplication of matrices.
1721. Functions with real values that are continuous on the interval \([-1, 1]\), under ordinary addition and multiplication of functions.
1722. Polynomials in the single unknown \( x \) with integral coefficients, under the ordinary operations of addition and multiplication.
1723. Polynomials in one unknown \( x \) with real coefficients, under the ordinary operations of addition and multiplication.
1724. All matrices of the form \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \) with rational or real \( a \) and \( b \), under the ordinary operations of addition and multiplication of matrices.
1725. Does the set \( \{ a_0 + \sum_{n=1}^{n} (a_n \cos kx + b_n \sin kx) \} \) with real coefficients form...
1726. Is a ring formed by numbers of the form \( a + b \sqrt{2} \), where \( a \) and \( b \) are rational, under the ordinary operations (for the sake of definiteness, take the real value of the root)?

1727. Show that numbers of the form \( a + b \sqrt{3} + c \sqrt{4} \), where \( a, b, \) and \( c \) are rational, form a field, and show that each element of the field in the indicated form can be represented uniquely. Find the inverse of the number \( \frac{3}{\sqrt{2}} + 2 \sqrt{4} \) (we assume the real value of the root).

1728. Prove that numbers of the form \( a + b \sqrt{5} + c \sqrt{25} \) with rational \( a, b, \) and \( c \) form a field; find the inverse of \( \frac{3}{\sqrt{2}} - 3 \sqrt{5} - 3 \sqrt{25} \) in that field.

*1729. Let \( \alpha \) be a root of the polynomial \( f(x) \) of degree \( n \geq 1 \) with rational coefficients, the polynomial not being reducible over the field of rationals. Prove that numbers of the form \( a_0 + a_1 \alpha + a_2 \alpha^2 + \ldots + a_{n-1} \alpha^{n-1} \) with rational \( a_0, a_1, a_2, \ldots, a_{n-1} \) form a field, each element of the field capable of being written down uniquely in the indicated form. We say that this field is obtained by adjoining the number \( \alpha \) to the field of rational numbers.

*1730. In the field obtained by adjoining the root \( \alpha \) of the polynomial \( f(x) = x^3 + 4x^2 + 2x - 6 \) (problem 1729) to the field of rational numbers, find the inverse of \( \beta = 3 - \alpha + \alpha^2 \).

1731. Prove that all diagonal matrices, that is, matrices of the form

\[
\begin{pmatrix}
a_1 & 0 & 0 & \cdots & 0 \\
0 & a_2 & 0 & \cdots & 0 \\
0 & 0 & a_3 & \cdots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_n
\end{pmatrix}
\]

of order \( n \geq 2 \) with real elements form a commutative ring with zero divisors under the ordinary operations of addition and multiplication of matrices.

1732. Give examples of zero divisors in the ring of functions continuous on the interval \([-1, +1]\).

1733. Prove that in the ring of square matrices of order \( n \) over some field, nonsingular matrices, and they alone, are divisors of zero.

1734. Show that the pairs of integers \((a, b)\) with the operations given by the equalities

\[
(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2),
\]

\[
(a_1, b_1) (a_2, b_2) = (a_1a_2 + b_1b_2, b_1a_2)
\]

form a ring, and find all zero divisors of that ring.

1735. Prove that a field does not have divisors of zero.

1736. Prove that the equality \( az = ay \) for a given element \( a \) and for any elements \( z \) and \( y \) of a ring implies the equality \( z = y \) if and only if \( a \) is not a left divisor of zero.

1737. Show that matrices of order \( n \geq 2 \) with elements taken from a certain field, in which matrices all rows from the second onwards consist of zeros, form a ring, and that in the ring any nonzero element will be a right divisor of zero. What matrices in the ring are not left divisors of zero?

*1738. Show that in a ring with unity \( e \) the commutativity of addition follows from the other axioms of the ring.

1739. Having verified that the property of zero and zero divisors can be proved without making use of the commutativity of addition, prove that in a ring containing at least one element \( c \) that is not a zero divisor, the commutativity of addition follows from the other axioms of the ring.

1740. Give examples of rings of matrices of a special kind that have several right or several left units.

1741. Given an integer \( n \geq 0 \). Two integers \( a \) and \( b \) are said to be congruent modulo \( n \) (in symbols: \( a \equiv b \pmod{n} \), if their difference \( a - b \) is divisible by \( n \). (For \( n = 0 \) this means that \( a = b \); for \( n > 0 \), this means that when \( a \) and \( b \) are divided by \( n \), they yield the same remainder called the residue modulo \( n \).) Show that the set of all integers \( \mathbb{Z} \) can be split into classes of congruent numbers that do not have any elements in common. Let us define the addition and multiplication of classes in terms of the appropriate operations on their representatives, that is, if the numbers \( a, b, a + b \) and \( ab \) belong, respectively, to the classes \( A, B, C \) and \( D \), then we set \( A + B = C \) and \( AB = D \).

Prove that under such operations the set of classes is a ring (the ring of residues \( \mathbb{Z}_n \) modulo \( n \)).
*1742. Prove that a finite commutative ring without zero divisors and containing only one element is a field.

*1743. Show that the ring of residues modulo \( n \) (problem 1741) is a field if and only if \( n \) is prime.

1744. A square matrix is said to be a scalar matrix if its elements on the main diagonal are equal and the elements off the diagonal are zero. Show that scalar matrices of order \( n \) with real elements under ordinary operations form a field that is isomorphic to the field of real numbers.

1745. Show that matrices of the form \( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \), where \( a \) and \( b \) are real numbers, form a field that is isomorphic to the field of complex numbers.

1746. Prove that the field of matrices of the form \( \begin{pmatrix} b & a \\ 2b & a \end{pmatrix} \) with rational \( a \) and \( b \) (problem 1724) is isomorphic to the field of numbers of the form \( a + b \sqrt{2} \) also with rational \( a \) and \( b \).

*1747. Prove that the algebra of real matrices of the form

\[
\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & c & b & a \end{pmatrix}
\]

is isomorphic to the algebra of quaternions \( a + bi + cj + dk \).

1748. Prove that the algebra of matrices of the form

\[
\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}
\]

with real \( a, b, c, d \) and \( i = \sqrt{-1} \) is isomorphic to the algebra of quaternions \( a + bi + cj + dk \).

1749. Find all automorphisms (that is, isomorphic mappings onto themselves) of the field of complex numbers, which automorphisms leave the real numbers unchanged.

*1750. Prove that any number field contains the field of rational numbers as a subfield.

*1751. Prove that under any isomorphism of number fields the subfield of rational numbers is mapped identically. In particular, the field of rational numbers admits only an identical isomorphic mapping into itself.

*1752. Prove that an identical mapping is the isomorphic mapping of the field of real numbers into itself.

1753. Use problem 1752 to find all isomorphic mappings of the field of complex numbers into itself, which mappings carry the real numbers into real numbers.

1754. Prove that the minimal subfield of any field of characteristic zero is isomorphic to the field of rational numbers.

1755. Prove that the minimal subfield of any field with characteristic \( p \) is isomorphic to the field of residues modulo \( p \).

1756. Solve the system of equations

\[
x + 2z = 1, \quad y + 2z = 2, \quad 2x + z = 1
\]

in the field of residues modulo 3 and modulo 5.

1757. Solve the system of equations

\[
x + y + 2z = 1, \quad x + 2y + 3z = 1, \quad 4x + 3y + 2z = 1
\]

in the field of residues modulo 5 and modulo 7.

1758. Find the largest common divisor of the polynomials

\[
f(x) = x^3 + x^2 + 2x + 2, \quad g(x) = x^2 + x + 1
\]

(a) over the field of residues modulo 3;
(b) over the field of rational numbers.

1759. Find the largest common divisor of the polynomials

\[
f(x) = 5x^3 + x^2 + 5x + 1, \quad g(x) = 5x^2 + 21x + 4
\]

(a) over the field of residues modulo 5 (here, each coefficient \( a \) is to be understood as the multiple \( a \) of the unit of the indicated field or the coefficients are to be replaced by their smallest nonnegative residues modulo 5);
(b) over the field of rational numbers.

1760. Find the largest common divisor of the polynomials

\[
f(x) = x^3 + 1, \quad g(x) = x^3 + x + 1
\]

over the field of residues modulo (a) 3, (b) 5.

*1761. (a) Prove that if the polynomials \( f(x) \) and \( g(x) \) with integer coefficients are relatively prime over the field \( \mathbb{Z}_p \) of residues with respect to the prime modulus \( p \), and at least one of the leading coefficients is not divisible by \( p \),
then these polynomials are relatively prime over the field of rational numbers.

(b) Show by way of an example that for any prime \( p \) the converse assertion does not hold.

*1762. Prove that the polynomials \( f(x) \) and \( g(x) \) with integer coefficients are relatively prime over the field of rational numbers if and only if they are relatively prime over the field of residues modulo \( p \), where \( p \) is any prime, with the possible exception of a finite set of such numbers.

1763. Factor the polynomial \( x^4 + x^3 + x^2 + 1 \) into irreducible factors over the field of residues modulo 2.

1764. Factor the polynomial \( x^3 + 2x^2 + 4x + 1 \) into irreducible factors over the field of residues modulo 2.

1765. Factor the polynomial \( x^4 + x^3 + x + 2 \) into irreducible factors over the field of residues modulo 3.

1766. Factor the polynomial \( x^4 + 3x^3 + 2x^2 + x + 4 \) into irreducible factors over the field of residues modulo 5.

1767. Factor all polynomials of degree two in \( x \) into irreducible factors over the field of residues modulo 2.

1768. Factor all third-degree polynomials in \( x \) into irreducible factors over the field of residues modulo 2.

1769. Find all second-degree polynomials in \( x \) with leading coefficient 1 that are irreducible over the field of residues modulo 3.

1770. Find all third-degree polynomials in \( x \) with leading coefficient 1 that are irreducible over the field of residues modulo 3.

1771. Prove that if a polynomial \( f(x) \) with integer coefficients is reducible over the field of rational numbers, then it is reducible over the field of residues with respect to any prime modulus \( p \) that does not divide the leading coefficient. Give an example of a polynomial that is reducible over the field of rational numbers but is irreducible over the field of residues modulo \( p \), where \( p \) divides the leading coefficient.

1772. Prove that the multiplicative group \( G \) of the field \( Z_p \) of residues with respect to the prime modulus \( p \) is cyclic.

*1773. There exist polynomials with integer coefficients that are irreducible over the field of rational numbers but are reducible over the field of residues with respect to any prime modulus \( p \).

1774. Prove that if all the elements of a commutative ring \( R \) have a common divisor \( a \), then the ring has a unit element.

1775. Indicate a commutative ring with unity that contains the element \( a \neq 0 \) with one of the following properties:

(a) \( a^n = 0 \), (b) for a given integer \( n > 1 \) the following conditions hold: \( a^n = 0, a^k \neq 0 \) if \( 0 < k < n \). (1776. Let \( R \) be a commutative ring with unity \( e \). Prove that (a) the multiplicative inverse (that is, the divisor of unity) cannot be a divisor of zero;

(b) the multiplicative inverse has a unique inverse element;

(c) if \( a, b \) are inversible, then \( a \) is divisible by \( b \) if and only if \( ab \) is divisible by \( ba \);

(d) the principal ideal \( (a) \) of element \( a \) in \( R \) is different from \( R \) if and only if \( a \) is not inversible.

1777. Let \( R \) be a commutative ring with unity \( e \) and without divisors of zero. Prove that

(a) the elements \( a \) and \( b \) are associated if and only if each of them is divisible by the other;

(b) the principal ideals \( (a) \) and \( (b) \) coincide if and only if \( a \) and \( b \) are associated (the definition of a principal ideal is given in problem 1783).

1778. Let \( R \) be a commutative ring with unity \( e \) and let \( R (x) \) be the set of all formal power series \( \sum_{n=0}^{\infty} \alpha_n x^n \), \( \alpha_n \in R \).

We introduce the ordinary operations of addition and multiplication of series:

\[
\sum_{n=0}^{\infty} \alpha_n x^n + \sum_{n=0}^{\infty} \beta_n x^n = \sum_{n=0}^{\infty} (\alpha_n + \beta_n) x^n,
\]

\[
(\sum_{n=0}^{\infty} \alpha_n x^n)(\sum_{n=0}^{\infty} \beta_n x^n) = \sum_{n=0}^{\infty} \gamma_n x^n,
\]

where \( \gamma_n = \sum_{h=0}^{n} \alpha_h \beta_{n-h} \).
Show that
(a) \( R(x) \) is a commutative ring with unity;
(b) \( R(x) \) contains a subring isomorphic to \( R \);
(c) if \( R \) does not have zero divisors, then it also holds for \( R(x) \);
(d) if \( R \) is a field, then \( \sum_{n=0}^{\infty} a_n x^n \) is a multiplicative inverse of the ring \( R(x) \) if and only if \( a_0 \neq 0 \).

*1770. Let \( R \) be the set of all numbers of the form \( a + b \sqrt{-3} \), where \( a \) and \( b \) are rational integers. Show that \( R \) is a ring with unity in which factorization into prime factors exists but is not unique. In particular, show that in the two factorizations
\[
4 = 2 \times 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})
\]
the factors are prime, and 2 is not associated with 1 ± \( \sqrt{-3} \).

*1771. Prove that all finite sums \( \sum_{i=1}^{n} a_i x^i \) with real \( a_i \) and nonnegative dyadic-rational \( r_i \), with respect to the ordinary operations of addition and multiplication of functions form a commutative ring with unity and without zero divisors, in which ring there are no prime elements.

1781. Will the following sets be subgroups of the additive group, subrings, or ideals of the rings indicated below:
(a) the set \( n\mathbb{Z} \) of numbers that are multiples of \( n \geq 1 \), in the ring of integers \( \mathbb{Z} \);
(b) the set of integers \( Z \) in the ring \( \mathbb{Z} [x] \) of integral polynomials;
(c) the set \( n\mathbb{Z} [x] \) of polynomials whose coefficients are multiples of the number \( n \geq 1 \), in the ring \( \mathbb{Z} [x] \) of integral polynomials;
(d) the set \( N \) of natural numbers in the ring of integers \( \mathbb{Z} \);
(e) the set \( N \mathbb{Z} [x] \) of polynomials with even constant term in the ring \( \mathbb{Z} [x] \) of integral polynomials;
(f) the set \( Z [x] \) of polynomials with even leading coefficient in the ring \( \mathbb{Z} [x] \) of integral polynomials.

1782. Prove that the intersection of any set of ideals of a commutative ring \( R \) is an ideal.

1783. The principal ideal \( (a) \) generated by an element \( a \) of a commutative ring \( R \) is a minimal ideal containing \( a \). Prove that the ideal \( (a) \) exists for any element \( a \in R \) and consists of all the elements of the form
(a) \( ra \), where \( r \) is any element in \( R \) if \( R \) has unity;
(b) \( ra + ma \), where \( r \) is any element in \( R \) and \( m \) is any integer if \( R \) does not have a unit element.

1784. A minimal ideal containing \( M \) is called the ideal \( (M) \) generated by the set \( M \) of a commutative ring \( R \). If the set \( M \) consists of a finite number of elements \( a_1, \ldots, a_n \), then the ideal \( (M) \) is also denoted as \( (a_1, \ldots, a_n) \). Prove that the ideal \( (M) \) exists for any nonempty set \( M \subseteq R \) and consists of all finite sums of the form
(a) \( \sum r_i a_i ; r_i \in R, a_i \in \mathbb{M} \) if \( R \) has a unit element;
(b) \( \sum r_i a_i + \sum n_i a_i ; r_i \in R, a_i \in \mathbb{M} ; n_i \) are integers if \( R \) does not have a unit element.

*1785. A ring of principal ideals is a commutative ring with unity and without zero divisors, in which each ideal is a principal ideal (see problem 1783). Prove that each of the following rings is a ring of principal ideals:
(a) the ring of integers, \( \mathbb{Z} \);
(b) the ring \( \mathbb{F} [x] \) of polynomials in the single unknown \( x \) over the field \( \mathbb{F} \);
(c) the ring \( A \) of Gaussian integers.

1786. The sum of ideals \( I_1, I_2, \ldots, I_n \) of a commutative ring \( R \) is the set \( I \) of all elements \( x \) in \( R \) that can be represented as
\[
x = x_1 + x_2 + \ldots + x_n ; \quad x_i \in I_i \quad (i = 1, 2, \ldots, n)
\]
We write \( I = I_1 + I_2 + \ldots + I_n \). If for any \( x \) in \( I \), the indicated representation is unique, then the sum \( I \) is called the direct sum of ideals \( I_i \). In this case we write \( I = I_1 + I_2 + \ldots + I_n \).
Prove that
(a) the sum of any finite number of ideals is an ideal;
(b) the sum of two ideals is a direct sum if and only if their intersection contains zero alone.

1787. Prove that if \( I = I_1 + I_2 \) is a direct sum of the ideals \( I_1, I_2 \), then the product of any element of \( I_1 \) by any element of \( I_2 \) is equal to zero.

1788. Let \( R = I_1 + I_2 \) be a decomposition of a commutative ring \( R \) with unit \( e \) into a direct sum of nonzero ideals \( I_1, I_2 \).

Prove that if \( e = e_1 + e_2 \), \( e_1 \in I_1, e_2 \in I_2 \), then \( e_1, e_2 \) will be unit elements, respectively, in \( I_1, I_2 \), but not in \( R \).

1789. Prove that the factor ring of the ring \( D[x] \) of polynomials with real coefficients with respect to the ideal of polynomials divisible by \( x^2 + 1 \) is isomorphic to the field of complex numbers \( a + bi \) with the operations of addition and multiplication defined by the familiar school-textbook rules.

1790. Prove that any homomorphic mapping of a field \( P \) into a ring \( R \) is either an isomorphic mapping onto some field that enters into \( R \) as a subring (a so-called embedding of \( P \) in \( R \)), or a mapping of all elements of \( P \) into the zero of \( R \).

1791. Let \( Z \) be a ring of integers and let \( R \) be any ring with the unit element \( e \). Prove that the mapping \( \phi \), for which \( \phi(n) = ne \), is a homomorphic mapping of \( Z \) into \( R \). Find the image \( \phi(Z) \) of the ring \( Z \) under this homomorphism.

1792. Let \( A \) be the ring of Gaussian integers and let \( I \) be the set of all numbers \( a + bi \) with even \( a \) and \( b \); (a) show that \( I \) is an ideal in \( A \), (b) find the cosets of \( A \) with respect to \( I \), (c) in the factor ring \( A/I \) find the zero divisors and show in that way that \( A/I \) is not a field.

1793. Prove that the factor ring \( A/I \) of the ring of Gaussian integers, with respect to the principal ideal \( I = (3) \) is a field of nine elements.

1794. Prove that the factor ring \( A/I \) of the ring of Gaussian integers, with respect to the principal ideal \( I = (a) \) is a field if and only if \( a \) is prime and not equal to the sum of two squares of integers.

1795. Let \( P[x, y] \) be a ring of polynomials in two unknowns \( x, y \) over the field \( P \) and let \( I \) be the set of all polynomials of this ring without the constant term. Prove that

(a) \( I \) is an ideal but is not a principal ideal;
(b) the factor ring \( P \langle x, y \rangle/I \) is isomorphic to the field \( P \).

1796. Let \( I = (x, y) \) be an ideal, generated by a set of two elements \( x \) and \( y \), in the ring of integral polynomials \( Z[x] \). Prove that
(a) the ideal \( I \) consists of all polynomials with even constant terms;
(b) the ideal \( I \) is not a principal ideal;
(c) the factor ring \( Z[x]/I \) is isomorphic to the field of residues modulo 2.

1797. Let \( (n) \) be an ideal generated by the integer \( n > 1 \) in the ring of integral polynomials \( Z[x] \). Prove that the factor ring \( Z[x]/(n) \) is isomorphic to the ring \( Z_n[x] \) of polynomials over the ring of residues modulo \( n \).

1798. Let \( R \) be the ring of all real functions \( f(x) \) defined on the whole number line under the ordinary operations of addition and multiplication and let \( e \) be a real number. Prove that
(a) the mapping \( \phi \) of \( f(x) \) onto \( f(e) \) is a homomorphic mapping of the ring \( R \) onto the field \( D \) of real numbers;
(b) the kernel of a homomorphism \( \phi \), that is, the set \( I \) of all elements of the ring \( R \) that are mapped into the number 0, is an ideal in \( R \);
(c) the factor ring \( R/I \) is isomorphic to the field of real numbers \( D \).

1799. Let \( Z_p \) be a field of residues with respect to the prime modulus \( p \) and let \( f(x) \) be a polynomial of degree \( n \) in the ring \( Z_p[x] \) that is irreducible over the field \( Z_p \) (from field theory we know that such a polynomial exists for any prime \( p \) and for any natural \( n \)), let \( I \) be a principal ideal generated by the polynomial \( f(x) \) in the ring \( Z_p[x] \). Prove that the factor ring \( Z_p[x]/I \) is a finite field and find the number of its elements.

Sec. 22. Modules

An algebra over a ring \( R \) is an Abelian group \( M \) (usually written additively) for the elements of which we define multiplication by elements of \( R \) so that \( \lambda a \in M \) for arbitrary \( \lambda \in R, a \in M \), and the following conditions hold (they are similar to the properties of multiplication of a vector by a scalar for a linear space):

1. \( \lambda (a + b) = \lambda a + \lambda b \),
2. \( (\lambda + \mu) a = \lambda a + \mu a \),
3. \( (\mu a) = (\lambda \mu) a \).
where $\lambda, \mu \in R, a, b \in M$.

If, besides, the ring $R$ has a unit element $e$ and

4. $e a = a, a \in M$,

then $M$ is called a unital left module over $R$.

A submodule of a left module $M$ over a ring $R$ is a subgroup $A$ of the group $M$ for which $\lambda a \in A$ for arbitrary $\lambda \in R, a \in A$.

A mapping $\varphi$ of the left module $M$ onto the left module $M'$ over one and the same ring $R$ is said to be homomorphic if $\varphi (a + b) = \varphi a + \varphi b, \varphi (\lambda a) = \lambda \varphi a$ for arbitrary $a, b \in M, \lambda \in R$.

A reciprocal one-to-one and homomorphic mapping of a module $M$ onto a module $M'$ (over the same ring) is said to be isomorphic (or an isomorphism) and the modules $M$ and $M'$ are termed isomorphic.

A factor module $M/A$ of a left module $M$ over a ring $R$ with respect to the submodule $A$ is the factor group $M/A$ with ordinary multiplication by the elements of the ring $R$: $\lambda (x + A) = \lambda x + A$.

The order $O(a)$ of the annihilator Ann $(a)$ of an element $a$ of the left module $M$ over a ring $R$ is the set of all elements $\lambda \in R$ for which $\lambda a = 0$. If the order of the element $a$ contains only zero of the ring $R$, then $a$ is termed the free element (for the element of order zero). Otherwise $a$ is termed a periodic element (or an element of nonzero order).

We similarly define right modules $M$ over a ring $R$ with multiplication $a \lambda \in M$, $a \in M$, $\lambda \in R$, and the associated concepts.

When we speak in the following problems of a module $M$ over a ring $R$, then $M$ is, for the sake of definiteness, to be regarded as a left module over $R$, although the corresponding properties also hold true for a right module over $R$.

To simplify matters, some problems are stated for the case of a commutative ring, although they could be generalized to modules over noncommutative rings.

1803. Verify that

(a) if a commutative ring $R$ is regarded as a left module over itself, then the submodules of this module coincide with the ideals of the ring $R$;

(b) if a noncommutative ring $R$ is regarded as a left (or right) module over itself, then the submodules of this module coincide with the left (or right) ideals of the ring $R$.

1804. Show that an Abelian group $G$ that is primary with respect to the prime number $p$ (problem 1695) may be regarded as a unital module over the ring $R$ of rational numbers whose denominators are not divisible by $p$.

1805. A cyclic submodule generated by an element $a$ in the left module $M$ over a ring $R$ is a minimal submodule containing $a$. Prove that for any $a \in M$ the cyclic submodule $\{a\}$ exists and consists of all elements of the module $M$ having the form:

(a) $\lambda a$, where $\lambda \in R$ if $M$ is a unital module;
(b) $\lambda a + na$, where $\lambda \in R$ and $n$ is an integer if $M$ is any module.

1806. Prove that an $n$-dimensional linear space over a field $P$ is (under the same operations) a unital module over $P$ and this module can be decomposed into a direct sum of $n$ cyclic submodules.

1807. Let $M$ be a unital module over a commutative ring $R$ with unit element $e$, let $\{a\}$ and $\{b\}$ be cyclic modules, and let $O(a)$ and $O(b)$ be the orders, respectively, of $a$ and $b$.

(a) Prove that if $\{a\} = \{b\}$, then $O(a) = O(b)$;
(b) use an example to illustrate the conditions $O(a) = O(b)$ are insufficient for the equality $\{a\} = \{b\}$;
(c) prove that for the equality $\{a\} = \{b\}$ it is necessary and sufficient that $b = ca, a = \beta b$, where $a, \beta$ are certain elements in $R$;
(d) prove that for the equality $\{a\} = \{b\}$ it is necessary and sufficient that the following conditions hold: $b = ca$ where $a \in R$ and can be inverted with respect to the module $O(a)$; that is, the coset $a + O(a)$ is an invertible element of the factor ring $R/O(a)$. 

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1808. Prove that any submodule $A$ of the cyclic module $M = \{a\}$ over the ring of principal ideals of $R$ is itself cyclic.

1809. Let $R$ be the set of all infinite sequences of integers $\sigma = (a_1, a_2, \ldots)$ with addition and multiplication with respect to components. Verify that $R$ is a commutative ring with unity, that is, a cyclic module over itself. In this module, find the submodule that does not have a finite system of generators. This shows that the submodule of a cyclic module need not be cyclic, and the submodule of a finitely generated module need not be finitely generated.

1810. Let $M$ be a module over the commutative ring $R$.

(a) Prove that if $R$ does not have zero divisors, then the set $A = M$ of all periodic elements is a submodule of the module $M$;

(b) use examples to show that for a ring $R$ with zero divisors the preceding statement may be erroneous.

1811. Let $M$ be a module over the ring $R$ of principal ideals and let $a$ and $b$ be periodic elements of $M$ of orders $O(a) = (a)$, $O(b) = (b)$ and let $\delta$ be the greatest common divisor of $a$ and $b$. Prove that the element $a + b$ is also periodic of order $O(a + b) = (\gamma)$, and that $\gamma$ divides $\frac{ab}{\delta}$.

1812. A module $M$ is said to be periodic if all its elements are periodic. The module $M$ over the ring $R$ of principal ideals is said to be primary if the orders of all elements in $M$, being ideals in $R$, are generated by the powers of one and the same simple element $p$ in $R$. The module $M$ is said to be a direct sum of the system (which is not necessarily finite) of its submodules $M_i$ if each nonzero element in $M$ can be uniquely represented as a sum of a finite number of nonzero elements taken one at a time from certain $M_i$. Prove that any periodic module $M$ over the ring $R$ of principal ideals can be decomposed into a direct sum of primary submodules.

1813. Let $M$ be a module over a ring $R$. Prove the following theorem: in order that any submodule in $M$ can have a finite number of generators, it is necessary and sufficient that in $M$ the condition of maximality for submodules be satisfied: any increasing sequence of submodules (not necessarily distinct) $M_1 \subseteq M_2 \subseteq \ldots$ becomes stabilized at a finite stage. In particular, this holds for ideals of the ring $R$ if it is regarded as a module over itself.

1814. Prove that if $\{a\}$ and $\{b\}$ are unital cyclic modules over one and the same ring $R$ and the orders of $a$ and $b$ are related by the inclusion $O(a) \subseteq O(b)$, then there exists a homomorphic mapping of $\{a\}$ onto $\{b\}$.

1815. Let $M = \{a\}$ be a unital cyclic module over a commutative ring $R$ with unit element 1. Prove that:

(a) the order $O(a)$ of the element $a$ is an ideal in $R$;
(b) the factor ring $R/\langle O(a) \rangle$, regarded as a module over $R$ with ordinary multiplication defined by multiplication in $R$, is isomorphic to the module $M$.

1816. Let $M = A + B$ be a decomposition of a module $M$ over a ring $R$ into a direct sum of submodules $A$ and $B$. Prove that the factor module $M/A$ is isomorphic, as a module over $R$, to the module $B$.

1817. A module $M$ is said to be an extension of a module $A$ with the aid of a module $B$ if $A$ is a submodule of $M$ and the module $M/A$ is isomorphic to $B$. Prove that the extension of a finitely generated module with the aid of a finitely generated module is a finitely generated module.

1818. Prove that if the condition of maximality for submodules is fulfilled in a module $M$ (see problem 1813), then this condition is fulfilled in the factor module $M/A$ over the ring $R$. Prove the following theorem on isomorphism:

$$(A + B)/A \cong B/(A \cap B).$$

1820. Let $M$ be a unital module with a finite number of generators $x_1, x_2, \ldots, x_n$ over a commutative ring $R$ with unit element. Prove that if the condition of maximality is fulfilled for ideals in the ring $R$, then it is fulfilled for submodules in the module $M$ (this theorem can be generalized to noncommutative rings with the maximality condition for left or right ideals).
Sec. 23. Linear Spaces and Linear Transformations
(Appendices to Sections 10, 16-19)

1821. Prove that for the equality $\alpha x + \beta y = \beta x + \alpha y$, where $\alpha$, $\beta$ are numbers and $x$, $y$ are vectors, to be fulfilled, it is necessary and sufficient that either $\alpha = \beta$ or $x = y$ hold.

1822. (a) Without using the commutativity of addition of vectors, prove that the right inverse and zero elements will be left as well;
(b) using item (a), prove that the commutativity of addition of vectors follows from the other axioms of a linear space.

1823. Let $D$ be the field of real numbers and $V$ the set of all functions specified and assuming positive values on the interval $[a, b]$. We define the addition of two functions and the multiplication of a function by a scalar by the following equalities:

\[ f \oplus g = fg, \quad \alpha \odot f = f^\alpha, \quad f, g \in V, \quad \alpha \in D.\]

(a) Verify that under the indicated operations, $V$ is a linear space over the field $D$;
(b) prove that the space $V$ is isomorphic to the space $V^*$ of all real functions specified on the interval $[a, b]$ under the ordinary operations of addition of functions and multiplication of a function by a real number;
(c) find the dimension of the space $V$.

1824. Prove the linear independence of the set of functions $x^\lambda_1, \ldots, x^\lambda_n$, where $\lambda_1, \ldots, \lambda_n$ are pairwise distinct real numbers.

1825. Prove the linear independence of the set of functions $x_1^\alpha_1, \ldots, x_n^\alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are pairwise distinct real numbers.

1826. Prove the linear independence of the following sets of functions:
(a) $1, \sin x, \cos x$;
(b) $1, \sin x, \cos x$;
(c) $\sin x, \sin 2x, \ldots, \sin nx$;
(d) $1, \cos x, \cos 2x, \ldots, \cos nx$;
(e) $1, \sin x, \sin 2x, \sin 3x, \ldots, \sin nx$.

1827. Prove the linear independence of the following sets of functions:
(a) $1, \sin x, \sin^2 x, \ldots, \sin^n x$;
(b) $1, \cos x, \cos^2 x, \ldots, \cos^n x$.

1828. Prove the linear dependence of the following sets of functions:
(a) $1, \sin x, \cos x, \sin^2 x, \cos^2 x, \ldots, \sin^n x, \cos^m x$ for $n > 2$;
(b) $1, \sin x, \cos x, \sin^3 x, \cos^3 x, \ldots, \sin^n x, \cos^n x$ for $n > 4$.

1829. Verify that all homogeneous polynomials of degree $k$ in $n$ unknowns $x_1, x_2, \ldots, x_n$ with real coefficients (or with coefficients taken from any field) together with zero under ordinary operations form a linear space; find the dimension of the space.

1830. Verify that all polynomials of degree $\leq k$ in $n$ unknowns $x_1, x_2, \ldots, x_n$ with real coefficients (or coefficients from any field) together with zero form a linear space under ordinary operations; find the dimension of that space.

1831. Let $V$ be a linear space of all polynomials in $x$ of degree $\leq n$, $n > 1$, with real coefficients.
(a) Prove that the set $L$ of all polynomials in $V$ having the given real root $c$ is a subspace of $V$;
(b) find the dimension of $L$;
(c) the same for the set $L_k$, $1 < k < n$, of all polynomials in $V$ having $k$ distinct real roots $c_1, \ldots, c_k$ (not counting multiplicity);
(d) is the set $L'$ of all polynomials in $V$ having the simple real root $c$ a subspace?

1832. Prove the theorem: for two linearly independent systems with the same number of vectors,

\[ \begin{align*}
  x_1, \ldots, x_k \\
  y_1, \ldots, y_k
\end{align*} \tag{1, 2} \]

of $n$-dimensional space $V_n$ to be equivalent (or to generate one and the same subspace), it is necessary and sufficient that in any basis the respective minors of the matrices $A$ and $B$ made up of the coordinate rows of the vectors of the systems be proportional.

1833. Prove that any subspace $L$ of the $n$-dimensional linear space $V_n$ is a domain of the values of a certain linear transformation $q$.

1834. Prove that any subspace $L$ of the $n$-dimensional linear space $V_n$ is the kernel of some linear transformation $q$. 
if $q$, and $q_s$ are projections defined in item (b) and the pairwise distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of a linear transformation $\varphi$ we take a linearly independent system of eigenvectors, then the system containing all the vectors taken is linearly independent.

*1836. Let $V$ be a real linear space (or a linear space over a field $F$ whose characteristic is different from two) and let $V = L + M$ be a decomposition of the space $V$ into a direct sum of the subspaces $L$ and $M$. Then any vector $x$ can be uniquely represented in the form

$$l x = y + z; \quad y \in L, \quad z \in M.$$ A linear transformation $\varphi$ defined by the condition $\varphi x = y$ is called a projection of the space $V$ on $L$ parallel to $M$. According to problem 1358, projections coincide with idempotent transformations, that is with linear transformations having the property $\varphi^2 = \varphi$. Using this fact, prove the following statements:

(a) if $\varphi$ is a projection on $L$ parallel to $M$ and $e$ is the identical transformation, then $e - \varphi$ is a projection on $M$ parallel to $L$;

(b) for the sum $\varphi = q_1 + q_2$ of two projections $q_1$ and $q_2$ to be a projection, it is necessary and sufficient that

$$\varphi q_2 = q_2 \varphi = \varphi,$$

(1)

where $\varphi$ is the zero transformation; if $q_1$ and $q_2$ are respectively the projections on $L_1$ parallel to $M_1$ and on $L_2$ parallel to $M_2$ with condition (1) holding, then $\varphi = q_1 + q_2$ is a projection on $L = L_1 \oplus L_2$ parallel to $M = M_1 \cap M_2$;

(c) for the difference $\varphi = q_1 - q_2$ of two projections $q_1$ and $q_2$ to be a projection, it is necessary and sufficient that

$$\varphi q_2 = q_2 \varphi = \varphi;$$

(2)

If $q_1$ and $q_2$ are projections defined in item (b) and the condition (3) holds, then $\varphi = q_1 q_2$ is a projection on $L = L_1 \oplus L_2$ parallel to $M = M_1 \cap M_2$.

1838. Find the distance between two planes $P_1: x = a_0 + t_1 a_1 + t_2 a_2$ and $P_2: x = b_0 + t_1 b_1 + t_2 b_2$, where $a_0 = (2, 1, 0, 1), a_1 = (1, 1, 1, 1), a_2 = (1, 0, 0, 1), b_0 = (1, -1, -1, 0), b_1 = (1, 1, 0, -1), b_2 = (1, 1, 2, 3)$, and the coordinates of the vectors are given in an orthonormal basis.

1839. A Hilbert space is a set $V$ of all infinite sequences of the real numbers $x = (x_1, x_2, \ldots)$ for which the series of squares $\sum_{i=1}^{\infty} |a_i|^2$ converges. Such sequences are called vectors (or points) of the space $V$. Addition of vectors, multiplication of a vector by a scalar, and scalar multiplication of vectors are defined in the usual fashion. Namely, if $x = (a_1, a_2, \ldots)$ and $y = (b_1, b_2, \ldots)$ are vectors and $\alpha$ is a scalar, then

$$x + y = (a_1 + b_1, a_2 + b_2, \ldots), \quad \alpha x = (\alpha a_1, \alpha a_2, \ldots).$$

Prove that

(a) $V$ is an infinite-dimensional Euclidean space;

(b) if $V^* = \{ \varphi: \varphi(x) = \langle \varphi, x \rangle \}$ is the orthogonal complement to the subspace $L$ in $V$ (problem 1365) then the equality $V = L \oplus L^*$ holds true for a finite-dimensional $L$;

(c) show with examples that the equality $L^* = L^*$ and $|L^*|^* = L^*$ are problems 1365 and 1366 may not hold for infinite-dimensional subspaces in $V$. 

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1840. Let \( V_n = L_1 + L_2 \) be a decomposition of an \( n \)-dimensional Euclidean space into a direct sum of two subspaces: \( L^*_1 \) and \( L^*_2 \), which are orthogonal complements of \( L_1 \) and \( L_2 \) respectively, and let \( q \) be a reflection of \( V_n \) in \( L_1 \) parallel to \( L_2 \). Prove that the transformation \( q^* \), the conjugate of \( q \), is a reflection of \( V_n \) in \( L^*_2 \) parallel to \( L^*_1 \).

1841. Find all isometric (or orthogonal) transformations that hold the zero vector fixed; (a) in the plane; (b) in three-dimensional space.

*1842. Find the geometric meaning of a linear transformation \( q \) of three-dimensional Euclidean space specified in the orthonormal basis \( e_1, e_2, e_3 \) by the matrix

\[
A = \begin{pmatrix}
3 & 4 & \sqrt{6} \\
4 & 3 & -\sqrt{6} \\
\sqrt{6} & -\sqrt{6} & 4 \\
4 & 4 & \sqrt{6}
\end{pmatrix}
\]

*1843. Determine the geometric meaning of a skew-symmetric transformation \( q \) of Euclidean space for the cases: (a) of a straight line; (b) of a plane; (c) of three-dimensional space. Show that in three-dimensional space, \( q \) reduces to the vector premultiplication of all vectors by one and the same vector \( a \), that is, \( qx = a \times x \).

*1844. Prove the following assertion: for a linear transformation \( q \) of Euclidean space (not necessarily finite-dimensional) to be skew-symmetric, it is necessary and sufficient that it carry each vector into the vector orthogonal to it.

Sec. 24. Linear, Bilinear and Quadratic Functions and Forms (Appendix to Sec. 15)

*1845. Prove that for any nonzero linear function \( l(x) \) specified in an \( n \)-dimensional linear space \( V_n \), there exists a canonical basis in which this function can be written in canonical form \( l(x) = x_1 \), where \( x_1 \) is the first coordinate of the vector \( x \) in that basis.

1846. Prove that the nonzero bilinear form \( b(x, y) = \sum_{j=1}^{n} a_j x_j y_j \) factors into a product of two linear forms

\[
b(x, y) = l_1(x) l_2(y), \quad \text{where } l_1(x) = \sum_{j=1}^{n} a_j x_j, \quad l_2(y) = \sum_{j=1}^{n} a_j y_j.
\]

1847. Prove that the bilinear form \( b(x, y) \) given in a real \( n \)-dimensional space (or in an \( n \)-dimensional space with a field of characteristic not equal to two) is symmetric if and only if it has a canonical basis in which it can be written by a bilinear form of canonical aspect: \( b(x, y) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \).

*1848. Prove that if the product of two linear function specified on a linear space \( V \) (not necessarily finite-dimensional) is identically zero, that is, \( l_1(x) l_2(x) = 0 \) for any \( x \in V \), then at least one of the functions is identically zero.

*1849. Prove that if the symmetric bilinear function \( b(x, y) \) specified in a linear space \( V \) (not necessarily finite-dimensional) decomposes into two linear functions \( b(x, y) = l_1(x) l_2(y) \), then it can be expressed as \( b(x, y) = \lambda l_1(x) l_2(y) \), where \( \lambda \) is a nonzero number and \( l_1(x) \) is a linear function.

1850. Prove that a bilinear function in an \( n \)-dimensional real space is of rank 1 if and only if, in some basis, it can be written as a form with the aspect

\[
(1) \pm x_1 y_1 \quad \text{for the function is symmetric;}
\]

\[
(2) x_1 y_2 \quad \text{for the function is nonsymmetric.}
\]

*1851. Prove that a nonzero skew-symmetric function on a linear space \( V \) (not necessarily finite-dimensional) cannot decompose into a product of two linear functions.

*1852. Let \( l(x) \) be a nonzero linear function on a linear space \( V \) (not necessarily finite-dimensional) cannot decompose into a product of two linear functions. Prove that

(a) the kernel \( S \) of the function \( l(x) \), that is, the set of all vectors \( x \in V \) for which \( l(x) = 0 \), is a maximal linear subspace, that is, \( S \) is not contained in a subspace \( T \) different from \( S \) and \( V \);

(b) for any vector \( a \) not in \( S \), any vector \( x \) can be uniquely represented as \( x = y + a \), where \( y \in S \).

*1853. Prove that if two linear functions \( l(x) \) and \( l(x) \) on a linear space \( V \) (not necessarily finite-dimensional) have the same kernel \( S \), then \( l(x) = \lambda l_1(x) \), where \( \lambda \) is a nonzero number.

*1854. Using the Jacobi method for computing minors, determine the affine class of surfaces in space.
dimensional space:

(a) \( x_1^2 + 2x_2^2 - x_3^2 + 2x_4x_5 + 2x_1 + 2x_2 = 0 \);
(b) \( x_1^2 + x_2^2 + 2x_3x_6 + 2x_4 + 2x_1 + 1 = 0 \).

1855. Prove that if \( n \) quadratic form with matrix \( A \) is positive definite, then so is the quadratic form
(a) with inverse matrix \( A^{-1} \);
(b) with the adjoint \( A^* \).

*1856. Let \( f(x) \) be a quadratic function on an \( n \)-dimensional real linear space \( V_n \). The vector \( x_0 \) is said to be a null vector if \( f(x_0) = 0 \). Prove that if the function \( f(x) \) is an alternating function, that is, there exist vectors \( x_1, x_2, \ldots, x_n \) such that \( f(x_1) > 0, f(x_2) < 0 \), then there is a basis consisting of null vectors. Suggest a method for constructing such a basis.

*1857. A null cone of a quadratic function \( f(x) \) is a set \( K \) of all null vectors (problem 1856). Prove that the null cone of a quadratic function \( f(x) \) on an \( n \)-dimensional real space \( V_n \) is a subspace if and only if \( f(x) \) is an alternating function, that is, either \( f(x) \) is positive definite or \( f(x) \) is negative definite.

*1858. Let \( f(x) \) be a quadratic function on an \( n \)-dimensional real linear space \( V_n \), let \( r \) be the rank, and \( p \) and \( q \) the positive and negative indices of inertia of that function. Prove that the maximal dimension of the linear subspaces that enter into the null cone \( K \) (problem 1857) is equal to

\[ \min(p, q) \text{ if } f(x) \text{ is nonsingular (that is, } r = n); \]

\[ n - \max(p, q) = \min(p, q) + n - r \text{ if } f(x) \text{ is arbitrary (singular or nonsingular).} \]

*1859. Let \( f(x) \) be a quadratic function with the same properties as those in the preceding problem. Prove that the maximal dimension of the linear manifold \( P \) that enters into the second order (quadratic) surface \( S \) given by the equation \( f(x) = 1 \) is equal to

\[ \min(p - 1, q) \text{ if } f(x) \text{ is nonsingular;} \]

\[ \min(p - 1, q) + n - r = n - \max(p, q + 1) \text{ in the general case.} \]

1860. Use problems 1858 and 1859 to find the maximal dimension of linear manifolds contained in the following quadratic surfaces if the dimension of the space is not indicated, then it is assumed to be equal to the largest number-label of the coordinates; the dimension of the empty manifold is taken to be equal to \(-1\):

(a) \( x_1^2 + x_2^2 - x_3^2 = 1 \) (hyperboloid of one sheet);
(b) \( x_1^2 - x_2^2 - x_3^2 = 1 \) (hyperboloid of two sheets);
(c) \( x_1^2 - x_2^2 = 0 \);
(d) \( x_1x_2 = 1 \);
(e) \( x_1x_2 = 4 \) (in three-dimensional space);
(f) \( x_1x_2 = 0 \);
(g) \( x_1x_2 = 1 \) (in \( n \)-dimensional space);
(h) \( x_1^2 - x_2^2 = 1 \);
(i) \( x_1^2 - x_2^2 - x_3^2 = 1 \);
(j) \( x_1^2 + x_2^2 - x_3^2 = 1 \);
(k) \( x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1 \);
(l) \( \sum_{k=1}^{n} (-1)^{k-1} x_k^2 = 1 \).

*1861. The left kernel (or left null space) of the bilinear function \( b(x, y) \) specified on a linear space \( V \) is the set \( L_b \) of all vectors \( x \in V \) for which \( b(x, y) = 0 \) for all \( y \in V \). The right kernel \( L_b^* \) is defined similarly.

Prove that (a) the left and right kernels are subspaces; (b) in an \( n \)-dimensional space the left and right kernels have the same dimension \( n - r \), where \( r \) is the rank of \( b(x, y) \), that is, the rank of its matrix with respect to some basis.

1862. Find the basis of the left and right kernel (null space) \( L_b \) and \( L_b^* \) (problem 1861) for the bilinear form \( b(x, y) = x_1y_1 + 2x_2y_2 + 3x_3y_3 + 6x_4y_4 \) and show that \( L_b \neq L_b^* \).

*1863. (a) Prove that for a symmetric and a skew-symmetric bilinear function the left kernel coincides with the right kernel;
(b) give an example of a bilinear function in \( n \)-dimensional space which is neither symmetric nor skew-symmetric but for which the left kernel coincides with the right kernel.

*1864. Prove that a nonzero skew-symmetric bilinear function can be represented in three-dimensional space as \( b(x, y) = a(x)b(y) - a(y)b(x) \), where \( a(x) \) and \( b(x) \) are linear functions.
1865. Let \( b(x, y) \) be a bilinear function and let \( L \) be a linear subspace of an \( n \)-dimensional space \( V_n \). Denote by \( L^* \) the set of all vectors \( y \in V_n \) such that \( b(x, y) = 0 \) for all \( x \in L \). Prove that:
(a) \( L^* \) is a subspace;
(b) if \( b(x, y) \) is nonsingular (that is, the rank is equal to \( n \)), then the dimension of \( L^* \) is equal to \( n - k \);
(c) if \( b(x, y) \) has a rank of \( r \leq n \), then the dimension of \( L^* \) is greater than or equal to \( n - k \).

1866. Let \( b(x, y) \) be a nonzero skew-symmetric bilinear function in an \( n \)-dimensional linear space \( V_n \). Prove that there is a basis in which \( b(x, y) \) can be written by a bilinear form of the following canonical form:
\[
b(x, y) = x_1 y_1 - x_2 y_2 + x_3 y_3 - \cdots - x_n y_n
\]
\[
[-E_{2k-1}y_{2k-1} - x_{2k-1}y_{2k-1}; 1 \leq k \leq \frac{n}{2}].
\]

For the following forms find the canonical aspect of the skew-symmetric bilinear form (problem 1866) and the nonsingular transformation of the unknowns that leads to it.

1867. \( b(x, y) = x_1 y_1 - x_2 y_2 + 2(x_3 y_3 - x_4 y_4) - x_5 y_5 + x_6 y_6 - x_7 y_7 \).

1868. \( b(x, y) = x_1 y_2 - x_2 y_1 + 2(x_3 y_3 - x_4 y_4) + 4(x_5 y_5 - x_6 y_6) \).

1869. Let \( f(x) \) be a quadratic function in an \( n \)-dimensional Euclidean space \( V_n \). Prove the following assertion:
for the cone \( K \) with equation \( f(x) = 0 \) (problem 1857) it contains an orthonormal basis of the space \( V_n \), it is necessary and sufficient that the trace of the matrix \( A \) of the function \( f(x) \) in any (and, hence, in any) orthonormal basis be equal to zero. Formulate this statement in matrix terms.

Sec. 25. Affine (or Point-Vector) Spaces

Definition. Let there be given a set \( A \) of the elements \( A, B, C \), and of ordered pairs of points and a linear space \( V \) (over the field of real numbers or over any field \( F \) with elements \( x, y, z \)), called vectors. Furthermore, let each ordered pair of points \( A, B \) (distinct or coincident) be associated with a unique vector \( x = AB \); two axioms hold relative to this correspondence:

(i) for any point \( A \) and any vector \( x \) there is a unique point \( B \) such that \( AB = x \);
(ii) for any (not necessarily distinct) three points \( A, B, C \) there is an equality that holds true: \( AB + BC = AC \).

The set \( A \) together with such a correspondence is termed an affine space.

If \( V = V_n \) is an \( n \)-dimensional linear space, then \( A \) is also an \( n \)-dimensional affine space and is denoted as \( A_n \). If \( V \) is infinite-dimensional, then \( A \) too is said to be infinite-dimensional. If the linear space \( V \) is Euclidean, then the affine space \( A \) is also termed Euclidean. In this case, the distance between the points \( A \) and \( B \) is equal to the length of the vector \( AB \) and the angle \( ABC \) is equal to the angle between the vectors \( BA \) and \( BC \).

Note. Any linear space \( V \) may be regarded as an affine space. Then the set \( A \) coincides with \( V \) so that the vectors are regarded as points too. We also say that the vector specifies a certain point of affine space. The association of an ordered pair of points of the vector (given in the definition) consists, in this case, in the fact that a vector \( x = y - x \) is associated with an ordered pair of points \( x, y \) in \( V \) whence, via \( x \) and \( z \), we can uniquely determine \( y \); this proves axiom I. Axiom II reduces to the obvious equality \( (y - x) + (z - y) = z - x \). This is the identification of points and vectors that is assumed in Secs. 18 to 19.

The plane of an affine space \( A \) passing through a point \( A \) and having as its direction subspace \( L \) is the set of all points \( M \) in \( A \) for which the vector \( AM \) belongs to \( L \).

The dimension of the plane \( \pi \) is the dimension of the affine subspace \( L \). The one-dimensional plane is called a straight line, and an affine linear plane of two-dimensional space is termed a plane.

Two planes \( \pi_1 \) and \( \pi_2 \) are said to be parallel if they do not intersect at any point in \( V \). In no plane in common and the common vector \( AB \) of them is contained in the directrix of each of the two coincides.
If a certain initial point \( O \in \mathbb{A} \) is chosen, then any point \( M \) is uniquely determined by the vector \( \overrightarrow{OM} \), and conversely. The vector \( \overrightarrow{OM} \) is termed the radius vector of the point \( M \). The plane \( \pi \) that passes through the point \( A \) and has the director subspace \( L \) consists of all points \( M \) whose radius vectors are determined from the equation \( \overrightarrow{OM} = \overrightarrow{OA} + x \), where \( x \in L \). If the vectors in \( V \) are regarded as points, then the plane is determined from the equality \( \pi = x_0 + L \), where \( x_0 = \overrightarrow{OA} \). Thus, in this case the concept of a plane coincides with that of a linear manifold given in Sec. 16. The planes passing through the point \( O \) will coincide with the subspaces.

An affine system of coordinates in an \( n \)-dimensional affine space \( \mathbb{A}_n \), consists of the point \( O \in \mathbb{A}_n \), called the origin, and the basis \( e_1, e_2, \ldots, e_n \) of the appropriate linear space \( V_n \). The coordinates of the point \( M \in \mathbb{A}_n \) are the coordinates of its radius vector \( \overrightarrow{OM} \) in the given basis, that is, the numbers \( x_1, x_2, \ldots, x_n \), that satisfy the equality

\[
\overrightarrow{OM} = x_1e_1 + x_2e_2 + \ldots + x_ne_n.
\]

Let a \( k \)-dimensional plane \( \pi \) of a real \( n \)-dimensional affine space pass through a point \( A \) with coordinates \( x_1^0, x_2^0, \ldots, x_n^0 \) and let it have a director subspace \( L \) with a basis consisting of vectors specified by their coordinates:

\[
e_i = (c_1^i, c_2^i, \ldots, c_n^i) \quad (i = 1, 2, \ldots, k).
\]

Then the coordinates of any point \( M \in \pi \) are given by

\[
x_i = x_i^0 + t_1c_1^i + \ldots + t_kc_k^i \quad (i = 1, 2, \ldots, n).
\]

These equations are called the parametric equations of the plane \( \pi \). The parameters \( t_1, t_2, \ldots, t_k \) assume arbitrary real values.

The same plane \( \pi \) may be specified by \( n-k \) linearly independent equations of the type

\[
\sum_{i=1}^{n} a_{ij}x_j = b_i \quad (i = 1, 2, \ldots, n-k).
\]

Here, \( b_i = \sum_{j=1}^{n} a_{ij}x_j^0 \) and the homogeneous equations

\[
\sum_{i=1}^{n} a_{ij}x_j = 0 \quad (i = 1, 2, \ldots, n-k)
\]

specify the subspace \( L \).

We will call the equations (2) the general equations of the plane \( \pi \).

The equations of a straight line passing through two points \( A(x_1^0, x_2^0, \ldots, x_n^0) \) and \( B(y_1^0, y_2^0, \ldots, y_n^0) \) are of the form

\[
x_i = x_i^0 + t(y_i^0 - x_i^0) \quad (i = 1, 2, \ldots, n).
\]

Here, \( t \) runs through all the real numbers.

A segment \( AB \) is a set of points \( M \) whose coordinates are obtained from (3) provided that \( 0 \leq t \leq 1 \). A point \( \Pi \) that divides \( AB \) in the ratio \( \lambda \neq -1 \) is determined in vector form by the condition \( \overrightarrow{AM} = \lambda \overrightarrow{MB} \) or in coordinates by

\[
x_i = \frac{x_i^0 + \lambda y_i^0}{1 + \lambda}, \quad i = 1, 2, \ldots, n.
\]

*1870. Given in an affine space four distinct points \( A, B, C, \) and \( D \). The points \( K, L, M, N \) divide the line segments \( AB, BC, CD, \) and \( DA \) in the same ratio of \( m/n \neq -1 \). Prove that

(a) if \( ABCD \) is a parallelogram, then \( KLMN \) is a parallelogram;
(b) if \( KLMN \) is a parallelogram and \( m \neq n \), then \( ABCD \) is also a parallelogram.

1871. Prove that a pair of coincident points is associated with a zero vector, that is, \( \overrightarrow{AA} = \overrightarrow{0} \).

1872. Prove that \( \overrightarrow{AB} = -\overrightarrow{BA} \).

1873. Prove that any plane \( \pi \) of affine space is itself an affine space whose dimension is equal to that of \( \pi \).

1874. Prove that the \( x \)-plane passing through a point \( A \) with director subspace \( L \) does not depend on the choice of \( A \) on it, that is, it coincides with the \( x' \)-plane passing through point \( A' \) from \( \pi \) with the same director subspace \( L \).

Find the parametric equations of the following planes that are specified by general equations.
1884. Using the notion of the rank of a matrix, all cases of the mutual positions of two two-dimensional planes in four-dimensional space specified by the equations

$$\sum_{j=1}^{5} a_{ij} x_j = b_i \quad (i = 1, 2)$$

and

$$\sum_{j=1}^{5} a_{ij} x_j = b_i \quad (i = 3, 4).$$

1885. Describe all cases of the mutual positions of two hyperplanes of n-dimensional affine space specified by the general equations

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = c$$

and

$$b_1 x_1 + b_2 x_2 + \ldots + b_n x_n = d.$$
1893. Express the condition of parallelism of two planes in $n$-dimensional affine space that are specified by general equations via the concept of the rank of a matrix.

1894. A hyperplane specified by the general equation \[ \sum_{i=1}^{n} a_i x_i = b \] partitions an $n$-dimensional affine space into two half-spaces consisting of points whose coordinates satisfy one of the inequalities \[ \sum_{i=1}^{n} a_i x_i > b \] or \[ \sum_{i=1}^{n} a_i x_i < b. \] Prove that each of these half-spaces is a convex set.

*1895. A polyhedron $P$ is given as a convex closure of a system of points of four-dimensional affine space specified by the coordinates:
\[ O(0, 0, 0, 0), A(1, 0, 0, 0), B(0, 1, 0, 0), C(1, 1, 0, 0), \]
\[ D(0, 0, 1, 0), E(0, 0, 1, 1), F(0, 0, 1, 1); \]
(a) write a system of linear inequalities specifying the polyhedron $P$;
(b) find all three-dimensional faces of the polyhedron.

1896. Solve the same problem as that of 1895 but for the points:
\[ O(0, 0, 0, 0), A(1, 0, 0, 0), B(0, 1, 0, 0), \]
\[ C(0, 1, 0, 0), D(1, 1, 0, 0), E(1, 0, 1, 0), \]
\[ F(1, 1, 1, 0), G(1, 1, 1, 1), H(0, 0, 0, 1). \]

1897. Find in three-dimensional space the vertices and form of a polyhedron $P$ given by the following system of inequalities:
\[ x_1 + x_2 + x_3 < -1, x_1 + x_2 + x_3 > -1, \]
\[ x_1 + x_3 < -1. \]

1898. Find the shape and vertices of sections of a four-dimensional cube specified in an orthonormal system of coordinates by the inequalities $x_i < 1$, $x_i > -1$, $i = 1, 2, 3, 4$.

Sec. 26. Tensor Algebra

What follows is a brief introduction to the notions and properties that are usually found in any lecture course on the subject. The proof of some of the properties is offered as problems in this section.

Suppose in an $n$-dimensional linear space $V_n$ (whether real, complex, or over any other field $F$) there are given two bases $e_1, e_2, \ldots, e_n$ and $e'_1, e'_2, \ldots, e'_n$. These bases are connected by the equalities
\[ e'_1 = c^1_1 e_1 + c^2_1 e_2 + \ldots + c^n_1 e_n, \]
\[ e'_2 = c^1_2 e_1 + c^2_2 e_2 + \ldots + c^n_2 e_n, \]
\[ \vdots \]
\[ e'_n = c^1_n e_1 + c^2_n e_2 + \ldots + c^n_n e_n, \]
or, more compactly,
\[ e'_i = c^i e_k \quad (i = 1, 2, \ldots, n). \]
We introduce the transition matrix
\[
C = \begin{pmatrix}
    c_{11} & c_{12} & \ldots & c_{1n} \\
    c_{21} & c_{22} & \ldots & c_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{n1} & c_{n2} & \ldots & c_{nn}
\end{pmatrix}
\]
in the columns of which are the coordinates of the vectors of the second basis with respect to the first basis. If the transition matrix is written in terms of rows (not columns), the formulas that employ matrix multiplication change, while the formulas (1) and (2) and others that do not use matrix multiplication remain unchanged. Then the formulas (1) can be written in matrix form by a single equation:

\[
(e_1', e_2', \ldots, e_n') = (e_1, e_2, \ldots, e_n)C.
\]

The coordinates of a vector \(x\) in the first basis will be expressed in terms of the coordinates of the same vector in the second basis via the rows of the matrix \(C\) using the formulas:

\[
x^k = c_{ik}x'^k, \quad k = 1, 2, \ldots, n.
\]

From this the coordinates \(x'^k\) can be expressed in terms of \(x^k\) in the form

\[
x'^k = d_{ik}x^k, \quad i = 1, 2, \ldots, n.
\]

We introduce the matrix
\[
D = \begin{pmatrix}
    d_{11} & d_{12} & \ldots & d_{1n} \\
    d_{21} & d_{22} & \ldots & d_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    d_{n1} & d_{n2} & \ldots & d_{nn}
\end{pmatrix}
\]
in the columns of which are the coefficients in the expressions \(x'^k\) in terms of \(x^k\). Then the formulas (2) can be written down in matrix form by the equality

\[
(x'^1, x'^2, \ldots, x'^n) = (x^1, x^2, \ldots, x^n)D.
\]

Since, on the other hand, \((x^1, x^2, \ldots, x^n) = (x'^1, x'^2, \ldots, x'^n)\), it follows that the matrices \(C\) and \(D\) are connected by the equality

\[
D = (C^*)^{-1}.
\]

From now on the asterisk denotes the transpose of a matrix.

If both matrices \(C\) and \(D\) are made up of the coefficients of formulas (1) and (2) not in columns but in rows, then these matrices become the transposes and the equality (3) remains unchanged.

The law of variation similar to that of the basis with respect to formulas (1) is called a covariant law, and the law associated with formulas (2) is called a contravariant law. Quantities (or other entities) associated with the basis and varying covariantly are termed covariant; they are denoted by lower indices; those varying contravariantly are termed contravariant and are denoted by upper indices.

A tensor in \(n\)-dimensional linear space is a correspondence under which to each basis of space there correspond \(n!\) numbers \(a_{11}a_{12}a_{13}\) labelled with \(p\) lower and \(q\) upper indices and varying with change of basis covariantly with respect to the lower indices and contravariantly with respect to the upper indices. Those numbers are called the components of the tensor in the given basis, and the number \(p + q\) is termed the rank or order of the tensor. We also say that a given tensor is \(p\)-fold covariant and \(q\)-fold contravariant, or of a tensor of type \((p, q)\).

By the definition of a tensor, its components in the two bases

\[
e_1, e_2, \ldots, e_n \quad \text{and} \quad e'_1, e'_2, \ldots, e'_n
\]

connected by (1) are themselves connected by the equality

\[
a'_{i_1j_1}a'_{i_2j_2} \ldots a'_{i_qj_q} = a_{i_1j_1}a_{i_2j_2} \ldots a_{i_qj_q}d_{j_1}d_{j_2} \ldots d_{j_q},
\]

\[
(l_1, \ldots, l_p, j_1, \ldots, j_q, j_q = 1, 2, \ldots, n).
\]

As usual, we assume summation from 1 to \(n\) over all indices \(k\) and \(l\).

A tensor may be defined differently as a geometric entity associated with a linear space \(V_1\). To do this we consider the conjugate space \(V^*_n\), its vectors are linear functions \(q(x)\) specified on the given space \(V_n\) under the ordinary operations of addition of two functions and the multiplication of a function by a scalar. The space \(V^*_n\) is also \(n\)-dimensional, and to each basis \(e_1, e_2, \ldots, e_n\) of the space \(V_n\) there corresponds a unique basis \(e'_1, e'_2, \ldots, e'_n\) of the conjugate space \(V^*_n\), called the conjugate (or dual) basis for the given basis of the space \(V_n\) and connected with it by the equalities

\[
e'_i(e_j) = \delta^i_j, \quad (i, j = 1, 2, \ldots, n),
\]

where \(\delta^i_j\) is the Kronecker delta equal to 1 when \(i = j\) and to 0 when \(i \neq j\).
If the basis $e_i$ transforms via formulas (1), then the conjugate basis $e^i$ transforms via the formulas

$$e^i = d^i_k e^k \quad (i = 1, 2, \ldots, n).$$

The following one-to-one correspondence can be established between the polylinear functions

$$F(x_1, x_2, \ldots, x_p), \quad \varphi_1, \varphi_2, \ldots, \varphi_q)$$

of $p$ vectors in $V_p$ and of $q$ vectors in $V^*_q$, on the one hand and all tensors of the type $(p, q)$ on $V_p$ on the other. This correspondence is isomorphic relative to the operations of addition, multiplication and multiplication by a scalar.

The above polylinear function there corresponds a tensor whose components in the basis $e_1, e_2, \ldots, e_n$ of the space $V_n$ are defined by the equalities

$$(e_{i_1}, e_{i_2}, \ldots, e_{i_p}), \quad \varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_q}) (6)$$

$$\varphi_{j_1} e_{i_1} + \varphi_{j_2} e_{i_2} + \cdots + \varphi_{j_q} e_{i_q} = F(e_{i_1}, e_{i_2}, \ldots, e_{i_p}, \varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_q})$$

Conversely, associated with a tensor having components $a_{i_1 \cdots i_p}$ in the basis $e_1, e_2, \ldots, e_n$ is a polylinear function defined by the equality

$$\sum_{i_1, i_2, \ldots, i_p} a_{i_1 \cdots i_p} e_{i_1} e_{i_2} \cdots e_{i_p}, \quad \varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_q}) (7)$$

where $a_{i_1 \cdots i_p}$ are components (or coordinates) of the vector $x_1, x_2, \ldots, x_p$ in the basis $e_1, e_2, \ldots, e_n$ and $\varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_q}$ are components of the vector $\varphi'1, \varphi'2, \ldots, \varphi'q$ in the conjugate basis $e'^1, e'^2, \ldots, e'^n$.

Here, the value of the polylinear function on the given vector does not depend on the choice of conjugate bases in $V_p$ and $V^*_q$.

In view of this correspondence, tensors on the space $V_p$ can be defined irrespective of the basis as polylinear functions on the vectors of the space $V_p$ and the conjugate space $V^*_q$.

A quantity that is a quadratic form (an inner-product space) in a vector space $V$ is a symmetric bilinear function $g(x, y)$ defined on $V$. A quadratic function if the quadratic function $g(x, x)$ is positive definite, then the space with such a metric is a Euclidean space and will be denoted as $E_n$. In this case, we will write $(x, y)$ instead of $g(x, y)$ and we will call the value of that function the scalar product (inner product) of the vectors $x$ and $y$.

In the space $M_n$, to each basis $e_1, e_2, \ldots, e_n$ there corresponds a unique dual basis $e^1, e^2, \ldots, e^n$ associated with the given basis by the equalities

$$g(e^i, e^j) = \delta^i_j \quad (i, j = 1, 2, \ldots, n).$$

Each vector $x$ of the space $M_n$ can be expanded in terms of the bases $e_i$ and $e^i$ with $x = x^i e_i = x_i e^i$. When the basis $e_i$ is changed by means of formulas (1) the components $x^i$ in that basis transform via formulas (2), that is, contravariantly, and are called contravariant components. The dual basis $e^i$ transforms via the formulas

$$e^i = d^i_k e^k \quad (i = 1, 2, \ldots, n)$$

and the components $x^i$ in this basis transform via the formulas

$$x^i = c^i_k x_k \quad (i = 1, 2, \ldots, n),$$

that is to say, covariantly, and are called covariant components.

Given a fixed vector $y \in M_n$, the function $\varphi_y(x) = g(x, y)$ is a linear function of $x$, that is, it is an element of the conjugate space $M^*_n$. The correspondence $y \rightarrow \varphi_y(x)$ is an isomorphic mapping of $M_n$ onto $M^*_n$. By identifying the elements of these spaces, which correspond to one another under the isomorphism, we may say that the space $M^*_n$, which is conjugate to the space with the quadratic metric $M_n$, coincides with $M_n$. In particular, this holds true for the Euclidean space $E_n$.

In the inner-product space $M_n$, one and the same polylinear function of $r$ vectors in $M_n$ may be regarded as a tensor of the type $(p, q)$, where $p + q = r$ and $p, q = 0, 1, 2, \ldots, r$. For this, choose one of the possible values for $p$, determine the components of the tensor in the given basis $e_i$ via formulas similar to (9), where $e^i$ is the dual basis connected with the basis $e_i$ by the formulas (1). Corresponding to the given bases of the type $(p, q)$, we define the value of the corresponding polylinear function on the given vector by a formula similar to (5). Here, the components $a_{i_1 \cdots i_p}$ in the formulas (6a) and (7) are to be understood as written in the same space $M_n$. \[208\]
In particular, to the metric function \( g(x, y) \) there correspond two tensors with components

\[
g^{ij} = g(e_i, e_j) \quad (i, j = 1, 2, \ldots, n),
\]

\[
g_{ij} = g(e^i, e^j) \quad (i, j = 1, 2, \ldots, n).
\]

which are called, respectively, the covariant metric tensor and the contravariant metric tensor. These tensors are of the type \((2,0)\) and \((0,2)\) respectively. Besides, the same function \( g(x, y) \) is associated with a third tensor of type \((1,1)\) whose components are given by the formulas (8) and do not depend on the choice of basis \( e_i \).

The values of the metric function \( g(x, y) \) are determined by one of the following three formulas:

\[
g(x, y) = g_{ij} x^i y^j, \tag{13}
\]

\[
g(x, y) = g^{ij} x_i y^j, \tag{14}
\]

\[
g(x, y) = x_i y^i. \tag{15}
\]

In particular, the scalar product \( x, y \) is computed in similar manner in Euclidean space.

Suppose, in Euclidean space, we consider only orthonormal bases. Then the metric \( C \) made up of the coefficients of formulas (1) is orthogonal, from (3) we obtain \( D = C \), the matrix made up of the coefficients of formulas (1) and (3) are transpose of each other, the dual basis coincides with the original basis, the covariant components coincide with the contravariant components, and the matrices of the metric tensors \( g_{ij} \) and \( g^{ij} \) turn into a single matrix.

In the case, all tensors of types \((p, q)\) that correspond coincide. The difference between the covariant and contravariant orders of the tensor is of no significance. And for that reason, all matrices can be written as subscripts while retaining the same coefficients.

Suppose, in Euclidean space \( E_n \), we have any basis \( e_1, e_2, \ldots, e_n \) of the covariant metric tensor in the given basis. Then the determinant of the matrix:

\[
g_{ij} = g(e_i, e_j)
\]

is the absolute value of the discriminant tensor, the oriented volume can be expressed by the formula

\[
V_{\sigma}(x_1, x_2, \ldots, x_n) = \sqrt{g} \cdot \det_{\sigma}(x_1, x_2, \ldots, x_n),
\]

where \( \sqrt{g} > 0 \) and \( \det_{\sigma}(x_1, x_2, \ldots, x_n) \) is the determinant made up of the coordinates (components) of the vectors \( x_1, x_2, \ldots, x_n \) in the given basis. Using the discriminant tensor, the oriented volume can be expressed by the formula

\[
V_{\sigma}(x_1, x_2, \ldots, x_n) = \varepsilon_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n},
\]

where \( \varepsilon \) is the 8th component of the vector \( x_i \) in the given basis.

An oriented volume becomes zero if and only if the vectors \( x_i \) are linearly dependent. For linearly independent \( x_i \) its absolute value is equal to the volume of the parallelepiped \( Q \), and it is a positive or negative depending on whether the set of vectors \( x_1, x_2, \ldots, x_n \) is identically or oppositely oriented with basis \( e_1, e_2, \ldots, e_n \).

A tensor product. Let \( \mathbf{V} \) and \( \mathbf{V} \) be two linear spaces over one and the same field \( P \). We consider ordered pairs of vectors, \( x = (x_1, x_2) \), \( x' = (x_1', x_2') \), and also the usual sum of such pairs: \( x + x' \). We now define the relation of the equivalence of the sums via the following rule: (1) Two sums that differ only in the sign of the summative are equivalent, i.e., the pairs \( (x_1 + x_2)', (x_1', x_2') \), where \( x = (x_1, x_2) \), are equivalent, or, the pair \( (x_1, x_2) + (x_1', x_2') \) is equivalent to the sum of the pairs \( x, x' + x, x' \).
Two sums are equivalent if we can pass from one of them to the other by applying the indicated rules to the whole sum or to a part of it a finite number of times. This relation of equivalence is reflexive, symmetric and transitive. For this reason, all sums can be separated into classes of equivalent sums. Let $T$ be the set of these classes. We introduce into $T$ the operations of addition and multiplication by elements of the field $P$ in terms of operations over the representatives of the classes. The sum of two sums is a sum obtained by adjoining to the first sum the terms of the second sum. Multiplication of a sum by $\alpha \in P$ is defined as the multiplication by $\alpha$ of the first elements of all pairs of the given sum. For example, $\alpha (ax' + yy') = (\alpha x) x' + (\alpha y) y'$. Under these operations, the set $T$ is a linear space over the field $P$. It is termed a tensor product of the spaces $V$ and $V'$ and is denoted as $V \times V'$. If $V$ and $V'$ are finite-dimensional, then $V \times V'$ is also finite-dimensional and its dimension is equal to the product of the dimensions of $V$ and $V'$. If $e_1, e_2, \ldots, e_n$ is a basis of $V$ and $e_1', e_2', \ldots, e_{n'}'$ is a basis of $V'$, then the system of classes of equivalent sums containing the ordered pairs

$$e_i e_j' \quad (i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n')$$

is a basis of the space $V \times V'$ (see problem 1918).

In similar fashion we can define a tensor product of any finite set of linear spaces.

Tensors of type $(p, q)$ specified on a space $V_n$ may be regarded as vectors of a space $T$, which is a tensor product $p$ of spaces $V_n$ and $q$ of the conjugate spaces $V_n^*$. To do this, we take the basis $e_1, e_2, \ldots, e_n$ in $V_n$ and the conjugate basis $e^1, e^2, \ldots, e^n$ in $V_n^*$. Then the ordered sets

$$e_{i_1} e_{i_2} \ldots e_{i_p} e^{j_1} e^{j_2} \ldots e^{j_q},$$

(20)

where all indices vary from 1 to $p$, constitute a basis of the space $T$. A vector $t$ in $T$ is expressed in terms of this basis as

$$t = a_{i_1 j_1} e_{i_1} e_{i_2} \ldots e_{i_p} e^{j_1} e^{j_2} \ldots e^{j_q}.$$

In components, that is, the numbers $a_{i_1 j_1}$, $a_{i_2 j_2}$, will be components of a $(p, q)$-type tensor in the basis $e_1, e_2, \ldots, e_n$, in the meaning of the first definition of a tensor.
a column and \( j \) to be the number of a row, we obtain a matrix made up of the components of the tensor:

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

Find the law of variation of the matrix \( A \) when passing to a new basis.

1907. (a) Show that the Kronecker delta \( \delta_{ij} \) yields a tensor of type \((1, 1)\) in all bases of an \( n \)-dimensional linear space; (b) write this tensor in the form of a polylinear function.

1908. Let \( a_{ij} \) be a twice covariant tensor in an \( n \)-dimensional space. Prove that

(a) if the matrix \( A \) of the tensor \( a_{ij} \) is nonsingular in one basis, then it is nonsingular in any basis;

(b) the elements \( a_{ij} \) of the inverse of \( A \) of the tensor \( a_{ij} \) (for the case when \( A \) is nonsingular) form a twice contravariant tensor.

1909. Let \( A \mathbf{x} \) be a linear transformation and let \( \varphi (\mathbf{x}) \) be a linear function on an \( n \)-dimensional linear space \( V_n \). Show that the function \( F(\mathbf{x}; \varphi) = \varphi(A\mathbf{x}) \) is a tensor of type \((1, 1)\), the matrix of which in any basis coincides with the matrix of the linear transformation \( A \mathbf{x} \) in the same basis.

1910. Show that the elements of the matrix of the bilinear function \( F(\mathbf{x}, \mathbf{y}) \) in an \( n \)-dimensional linear space form a tensor of type \((2, 0)\), that is to say, a twice covariant tensor.

1911. Prove that the elements of the matrix of a linear transformation in a given basis form a tensor of type \((1, 1)\).

1912. Given in an \( n \)-dimensional linear space \( V_n \), \( p \) vectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p \). Write down the coordinates (components) of the vectors in some basis according to the rows of the matrix

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

(a) Show that the numbers

\[
a_{i1} a_{j1} \quad a_{i1} a_{j2} \quad \cdots \quad a_{i1} a_{jn} \\
\]

are components of a \( p \)-fold contravariant tensor, that is, a tensor of type \((0, p)\) in the given basis. This tensor is termed a \( p \)-vector, (for \( p = 1 \) it is a vector, for \( p = 2 \) it is a bivector).

(b) Prove that a \( p \)-vector is zero if and only if the given vectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p \) are linearly dependent (in particular, when \( p > n \), all \( p \)-vectors are zero).

(c) Prove that two linearly independent sets, each made up of \( p \)-vectors, are equivalent if and only if the corresponding \( p \)-vectors differ in a nonzero factor.

(d) Show that if tensors are regarded as polylinear functions (see the introduction to this section), then a \( p \)-vector specified by the vectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p \) of the space \( V_n \) can be defined as a polylinear function of \( p \)-vectors \( \varphi^1, \varphi^2, \ldots, \varphi^p \) of the conjugate space \( V_n^* \) specified by the equation

\[
F(\varphi^1, \varphi^2, \ldots, \varphi^p) = \begin{vmatrix}
  \varphi^1(x_{11}) & \varphi^2(x_{11}) & \cdots & \varphi^p(x_{11}) \\
  \varphi^1(x_{21}) & \varphi^2(x_{21}) & \cdots & \varphi^p(x_{21}) \\
  \cdots & \cdots & \cdots & \cdots \\
  \varphi^1(x_{n1}) & \varphi^2(x_{n1}) & \cdots & \varphi^p(x_{n1})
\end{vmatrix}
\]

1913. Determine how the components of a tensor of type \((p, q)\) vary when passing from a basis \( e_1, e_2, \ldots, e_n \) to the basis \( e'_1, e'_2, \ldots, e'_n \) obtained from the preceding one by a given substitution \( \pi(i) = k_i \) (\( i = 1, 2, \ldots, n \)). This means that \( e'_i = e_{k_i} \) (\( i = 1, 2, \ldots, n \)).

1914. Relative to the basis \( e'_1, e'_2, \ldots, e'_n \), find the components of a tensor of type \((n, n)\) given by the equation

\[
F(x_1, \ldots, x_n; \varphi^1, \ldots, \varphi^n) = \begin{vmatrix}
  \varphi^1(x_1) & \varphi^2(x_1) & \cdots & \varphi^n(x_1) \\
  \varphi^1(x_2) & \varphi^2(x_2) & \cdots & \varphi^n(x_2) \\
  \cdots & \cdots & \cdots & \cdots \\
  \varphi^1(x_n) & \varphi^2(x_n) & \cdots & \varphi^n(x_n)
\end{vmatrix}
\]
1915. Let \( F(x_1, \ldots, x_n; \varphi^1, \ldots, \varphi^n) \) be a polylinear function that is skew-symmetric both in the arguments \( x_1, \ldots, x_n \) and in the arguments \( \varphi^1, \ldots, \varphi^n \). Prove that its values can be expressed in terms of the components in the given basis by the formula

\[
F(x_1, \ldots, x_n; \varphi^1, \ldots, \varphi^n) = \det (x) \det (\varphi) \cdot F(e_1, \ldots, e_n; e^1, \ldots, e^n),
\]

where \( e^1 \) is the conjugate of the basis \( e_1 \), \( \det (x) \) is the determinant made up of the components of the vectors \( x_1, \ldots, x_n \) in the basis \( e_1, \ldots, e_n \), and \( \det (\varphi) \) is the determinant made up of the components of the vectors \( \varphi^1, \ldots, \varphi^n \) in the basis \( e_1, \ldots, e^n \).

1916. Given a tensor of the type \( (p, q) \) in the form of a polylinear function \( F(x_1, \ldots, x_p; \varphi^1, \ldots, \varphi^q) \). Its contraction with respect to the numbers \( k \leq p, l \leq q \) is the sum

\[
\sum_{i_1, i_2, \ldots, i_p} F(x_1, \ldots, x_{k-1}, \varphi_{i_1}, \varphi_{i_2}, \ldots, x_p; \varphi^1, \ldots, \varphi^{l-1}, e^1, \\
\varphi^{l+1}, \ldots, \varphi^q).
\]

Show that the contraction does not depend on the basis and is a tensor of the type \( (p-1, q-1) \). It is assumed that \( p, q > 0 \).

1917. Given a symmetric tensor \( a_{ij} \) and a skew-symmetric tensor \( b_{ij} \). Find their complete contraction \( a_{ij} b^j \).

1918. In studying a tensor product (see the introduction to this section), it is convenient to use the concept of a contraction in the following meaning. For the sake of simplicity, let us consider the tensor product of two linear spaces: \( T = V \times V' \), where \( V \) and \( V' \) are linear spaces over the same field \( P \).

The contraction of the sum \( x_1 x'_1 + \ldots + x_k x'_k \), where \( x_j \in V, x'_j \in V' \) (\( i = 1, 2, \ldots, k \)), with the vector \( \varphi \) of the space \( V^* \), which is the conjugate of \( V \), is the vector \( a_{11} x'_1 + \ldots + a_{kk} x'_k \) \( \in V' \), where \( a_{ij} = \varphi (x_j) \in P \) (\( i = 1, 2, \ldots, k \)). Similarly we can define the contraction of the same sum with the vector \( \varphi' \in V'^* \); it is a vector in \( V'^* \).

1920. Prove that the tensor products of linear spaces of all polynomials in \( x \) and all polynomials in \( y \) with coefficients in the field \( P \) is a space of all polynomials in the two unknowns \( x, y \) with coefficients in \( P \).

1921. Prove the following statements:

(a) for any basis \( e_i \) of an \( n \)-dimensional Euclidean space \( E_n \) (or an inner-product space \( M_n \)) there is a unique dual basis \( e^i \) of the space \( E_n \) (respectively, of \( M_n \)) that is connected with the given basis by the equalities \( (e_i, e^j) = \delta^i_j \) (\( i, j = 1, 2, \ldots, n \)) [respectively, by the equalities \( g \) of the introduction to this section];

(b) the dual basis can be expressed in terms of the given basis via the formulas \( e^i = g^{ia} e_a \) (\( i = 1, 2, \ldots, n \));

(c) the given basis can be expressed in terms of the dual basis via the formulas \( e_i = g_{ia} e^a \) (\( i = 1, 2, \ldots, n \));

(d) the metric tensors \( g_{ij} \) and \( g^{ij} \) are connected by the equalities \( g_{ia} g^{ia} = \delta_i^i \) (\( i, j = 1, 2, \ldots, n \));

(e) if we denote the matrices of the metric tensors by \( G = (g_{ij}) \) and \( G^i = (g^{ij}) \), and the determinants of these matrices by \( g = |g_{ij}| \) and \( g^i = |g^{ij}| \), then \( GG^i = E \), \( g^i g = 1 \).

1922. Prove that the covariant and contravariant components of one and the same vector in a given basis of Euclidean space are connected by the following equalities:
1923. Two vectors are given in a rectangular Cartesian coordinate system:

\[ e_1 = (1, 0), \quad e_2 = (\cos \alpha, \sin \alpha), \quad \sin \alpha > 0; \]

(a) verify that these vectors form a basis;
(b) find the covariant metric tensor \( g_{ij} \) in the basis \( e_1, e_2 \);
(c) find the contravariant metric tensor \( g^{ij} \) in the basis \( e_1, e_2 \);
(d) find an expression for the vectors of the dual basis \( e^1, e^2 \) in terms of the basis \( e_1, e_2 \) and their components in the original rectangular system;
(e) write down an expression of the scalar product \((x, y)\) of two vectors in terms of their components in the basis \( e_1, e_2 \);
(f) find the discriminant tensor \( e_{ij} \) in the basis \( e_1, e_2 \);
(g) write down an expression for the oriented area of a parallelogram, constructed on the vectors \( x, y \), in terms of the discriminant tensor \( e_{ij} \) and in terms of the determinant made up of the components in the basis \( e_1, e_2 \).

*1924. Let \( e_1, e_2 \) be any basis of a plane, let \( g^{ij} \) be the contravariant metric tensor, and let \( e_{ij} \) be the discriminant in that basis. Using the given vector \( x = xe_1 \), one constructs a vector \( y = ye_2 \), where \( y^i = g^{ij}j_{jk} \). Determine to what extent the vector \( y \) depends on the choice of basis. What is the geometric relationship between the vectors \( x \) and \( y \)?

1927. A three-dimensional Euclidean space is defined by a metric tensor with the matrix:

\[
\begin{pmatrix}
1 & 3 & 1 \\
3 & 4 & 1 \\
1 & 1 & 6
\end{pmatrix}
\]

Find the lengths of line segments cut off on the coordinate axes by the plane \( x^1 + x^2 + x^3 = 1 \).

*1928. The metric of a three-dimensional Euclidean space is given by the metric tensor \( g_{ij} \) with matrix:

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

Find the area \( S \) of a triangle with vertices \( A(1, 0, 0), B(2, 1, 1), C(3, 1, 2) \) and height \( h \) dropped from \( C \) and \( AB \) if the coordinates and the tensor are given in the same basis.

*1929. A three-dimensional Euclidean space is given by the metric tensor \( g_{ij} \) with the matrix:

\[
\begin{pmatrix}
1 & 2 & 1 \\
2 & 5 & 3 \\
1 & 3 & 2
\end{pmatrix}
\]

Find the foot of the perpendicular \( P'Q \) dropped from the point \( P(1, -1, 2) \) on the plane \( x^1 + x^2 + 2x^3 + 2 = 0 \).

*1930. In a four-dimensional Euclidean space \( E_4 \) we are given three vectors \( x, y, z \) and we have constructed a vector \( u \) with covariant components \( u_{ij} = e_{ijkl}x^i y^k z^l (i, j, k, l = 1, 2, 3, 4) \), where \( e_{ijkl} \) is the discriminant tensor in the same basis (for a four-dimensional analogue of a vector product). Prove that:

(a) the vector \( u \) is orthogonal to each of the vectors \( x, y, z \);
(b) the vector \( r = x y z \) is immutably independent than the length \( \|r\| \) of the vector \( u \) is equal to the four-dimensional volume \( V(x, y, z) \) of a parallelepiped in \( x, y, z, u \); but if \( r, y, z \) are linearly dependent, then the volume is zero.

1931. In the completeness of the space \( E_4 \) we have six metric tensors of the type \( g_{ijkl} \) by the equations:

\[
\begin{align*}
g_{00} &= 1, & g_{01} &= 3, & g_{02} &= 1, & g_{03} &= 1, & g_{10} &= 1, & g_{11} &= 4, & g_{12} &= 1, & g_{13} &= 1, & g_{20} &= 1, & g_{21} &= 1, & g_{22} &= 2, & g_{23} &= 1, & g_{30} &= 1, & g_{31} &= 1, & g_{32} &= 1, & g_{33} &= 6,
\end{align*}
\]

Express \( g_{ij} \) in the matrix form.
(a) an oriented volume of the parallelepiped constructed
on the vectors \( \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \) (see the introduction to this
section) can be expressed by the equalities

\[ V(x_1, x_2, \ldots, x_n) = \sqrt{g_1 \det_1(x_1, x_2, \ldots, x_n)} \]

\[ = e_1 e_2 \cdots e_n x_1 x_2 \cdots x_n, \]

where \( \det_1(x_1, x_2, \ldots, x_n) \) is the determinant made up
of the covariant components of the vectors \( x_1, x_2, \ldots, x_n \)
and \( e_i \) is the \( i \)th covariant component of the vector \( x_i \); (b) \( \vec{e}_{i_1} \cdots \vec{e}_{i_n} = g_{i_1 \alpha_1} g_{i_2 \alpha_2} \cdots g_{i_n \alpha_n} \vec{e}_{\alpha_1} \vec{e}_{\alpha_2} \cdots \vec{e}_{\alpha_n} \)
(c) \( e_{i_1} \cdots e_{i_n} = g_{i_1 \alpha_1} g_{i_2 \alpha_2} \cdots g_{i_n \alpha_n} \vec{e}_{\alpha_1} \vec{e}_{\alpha_2} \cdots \vec{e}_{\alpha_n} \).

1933. Compute the contraction of the discriminant tensor
\( \epsilon_{ijk} \) of three-dimensional Euclidean space.
1934. In the basis \( e_1, e_2, e_3 \) of three-dimensional real
space we are given the covariant metric tensor \( g^{\mu} \) with the
matrix

\[ G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{pmatrix} \].

1935. Verify that the space is Euclidean;
find the matrix \( G_1 \) of the contravariant metric tensor.
find the contravariant components of the unit vector
in the normal to the plane given in the same basis by the
equation \( 2x^2 - 3x^3 + 5 = 0. \)
1936. Prove that the square of an oriented volume of
parallelepiped constructed on a vectors of \( n \)-dimensional
Euclidean space is equal to the determinant of the vectors.
1937. Given, in an \( n \)-dimensional Euclidean space, the
plane \( \pi \) specified by the equation \( a_1 x_1 + \cdots + a_n x_n = 0 \).
Also given is a point \( M(x_1, \ldots, x_n) \).
Show that the vector \( p \) with covariant components
\( p_{\alpha} = a_{\alpha} \)
is perpendicular to the plane \( \pi \).
Show that the distance of the point \( M \) to the
plane \( \pi \) can be expressed by the formula
\( d = \sqrt{g_{\alpha \beta} p^\alpha p^\beta}. \)
1937. Let the distance from the point \( M(x_1, x_2, \ldots, x_n) \)
on
to a plane to the straight line \( ax + by + cz = 0 \) be
expressed by the formula
\[ d = \frac{|ax_1 + by_1 + cz_1|}{\sqrt{a^2 + b^2 + c^2}}. \]
Chapter 1. Determinants

1. 1, 2, -2, 3, -4, 0, 3, 0, 0, -1, 7, 4ab, 8, -2b^2, 9, 1.
2. sin (a - b), cos (a + b), 12, 0, 13, 1, 14, 1, 15, -1, 16, 1.
3. 0, 18, 20, 0, 21, a^2 - b^2 + c^2 + d^2.
4. 22, y = -4; 23, x = 5; 24, y = 2.
5. 25, x = 3; 26, y = cos (b - c); 27, x = - cos a cos b; y = cos a sin b.
6. 28, The system is indeterminate; Cramer's formulas do not yield a correct answer since, by these formulas, x and y equal to 0, that is to say, they can assume arbitrary values, whereas they are connected by the relation 2x + 3y = 1, whence the value of one unknown determines a unique value of the other.
7. 29, The system is inconsistent.
8. 30, For a = b the equation is indeterminate, for a = b + c it is inconsistent, and for a = b - c it is indeterminate.
9. 31, For a = k, where k is an integer, the equation is inconsistent, and for all other values of a it is indeterminate.
10. 32, For a = 2k, where k is an integer, the equation is indeterminate, for a = (2k + 1) it is indeterminate, and for all other values of a it is indeterminate.
11. 33, For a + b = k, where k is an integer, the equation is indeterminate, for a + b = (2k + 1) it is indeterminate, and for all other values of a it is indeterminate.
12. 34, Solution. Suppose a given trinomial is a perfect square, that is, \(ax^2 + 2bx + c = (px + q)^2\). Comparing the coefficients of identical powers of \(x\), we find \(a = p^2\), \(b = pq\), \(c = q^2\), whence \(ac - b^2 = p^2q^2 - (pq)^2 = 0\). Suppose, conversely, \(ac - b^2 = 0\). Then \(ax^2 + 2bx + c = \frac{1}{a}(ax^2 + 2bx + ac) = \frac{1}{a}[(ax + b)^2 - (ac + b)^2] = \frac{1}{a}[(ax + b)^2 - x^2] = (ax + b)^2 - x^2 = 0\), where \(x = \frac{b}{a}\). Therefore \(ax + b = 0\), that is, \(a = b = 0\), and, assuming \(c = \frac{b}{a}\), we again have \(a = b = 0\).
13. 42. Solution. If \(ax + b = q\) for any \(x\), then \(ax + b = q(\epsilon + d)\), \(a = q\), \(b = q\) and \(ad - bc = 0\). Conversely, if \(ad - bc = 0\), then \(ax + b = q\). For \(c = 0, c = 0\) we have \(a = b\) and, assuming \(c = \frac{b}{a}\), we again have \(a = b = 0\). For \(c = 0, c = 0\) we get the same. Therefore \(ax + b = \frac{b}{a}\), which is a perfect square, since a square root may be taken of the complex number \(\frac{b}{a}\).
14. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. Divide each of the equations by abc and set \(\frac{a}{b} = x, \frac{b}{c} = y, \frac{c}{a} = z\). Divide each equation by abc and set \(\frac{a}{b} = x, \frac{b}{c} = y, \frac{c}{a} = z\).
81. \( x = \frac{a + b + c}{3}, \quad y = \frac{a + 2b + c}{3}, \quad z = \frac{a + b + 2c}{3}. \) Hint. This system can be solved via the Cramer formulas. It is simpler first to solve the second equation by \( z \) and the third by \( y \). Then, finally, solve the first equation by \( y \) and the second by \( x \). Use the relation \( 1 + e + e^2 = 0 \).

82. The system is indeterminate since the third equation is the sum of the other two and hence any solution of the first two equations satisfies the third one as well. The first two equations have an infinity of solutions, for example, \( x \) and \( y \) can be expressed in terms of \( z \) thus: \( x = 10z + 1, \ y = 7z \). We find the values of \( x \) and \( y \) by assigning an arbitrary value to \( z \).

83. The system is indeterminate.

84. The system is inconsistent since if for certain numerical values of the unknowns all the equations of the system became equalities, then by subtracting the first equality from the sum of the other two, we would again obtain an equality. But we obtain \( 0 = 4 \).

85. The system is inconsistent. For \( a^2 \neq 27 \) the system is evendetermined, and for \( a^2 = 27 \) it is inconsistent.

86. For \( a^2 \neq 27 \) the system is evendetermined, and for \( a^2 = 27 \) it is inconsistent.

87. For \( 4a^2 - 45a + 58 \neq 0 \) the system is evendetermined, and for \( 4a^2 - 45a + 58 = 0 \) it is inconsistent.

88. For \( ab \neq 15 \) the system is evendetermined, for \( a = 3, \ b = 5 \) it is indeterminate, and for \( ab = 15 \) but \( a = 3, \ b = 5 \) it is inconsistent.

89. For \( ab \neq 12 \) the system is evendetermined, for \( a = 3, \ b = 4 \) it is indeterminate, and for \( ab = 12 \) but \( a = 3, \ b = 4 \) it is inconsistent.

90. Hint. Consider a determinant in which the first two rows are not proportional (in particular none of the rows should contain zeros only), and the third row is equal to the sum of the first two, that is, each element is equal to the sum of the corresponding elements of the first two rows.

91. 0. 101. 0. 102. 0. 103. 0. 104. 0. 105. 0. 106. 0. 107. 0.

92. Two points \((x_1, y_1)\) and \((x_2, y_2)\) of the plane lie on a single straight line with the point dividing the segment between them in the ratio \(k:1\).

93. Hint. Add the second and third rows to the first and take advantage of the Vieta formula.

94. Hint. Add to the third column of the determinant of the left-hand side of the equation the second column multiplied by \( a + b + c \) and subtract the first multiplied by \( ab + bc + ca \).

95. 5. 124. 8. 125. 13. 126. 18. 127. \( \frac{n(n-1)}{2} \). 128. \( \frac{n(n+1)}{2} \).

96. 3\( n(n-1) \) inversions. The permutation is even for \( n \) equal to \( 4k + 1 \), \( 4k + 2 \), \( 4k + 3 \), where \( k \) is any nonnegative integer.

97. 3\( n(n-1) \) inversions. The permutation is even for \( n = 4k \), \( 4k + 3 \) and odd for \( n = 4k + 1, \ 4k + 2 \), where \( k \) is any nonnegative integer.

98. \( \frac{(3n+1)}{2} \) inversions. The permutation is even for \( n = 4k \), \( 4k + 1 \) and odd for \( n = 4k + 2, \ 4k + 3 \), where \( k \) is any nonnegative integer.

99. For \( ab = 12 \) the system is evendetermined, for \( a = 3, \ b = 4 \) it is inconsistent.

100. Solution. We take any two elements \( a_i, a_j \) in the given permutation \( i < j \).

101. In the given permutation these elements constitute an order, then in the original arrangement \( a_i \) comes before \( a_j \), and the indices \( i, j \) form an order. But if in the given permutation the elements \( a_i, a_j \) form an inversion, then in the original arrangement, \( a_j \) comes before \( a_i \), and so their indices \( j, i \) also form an inversion.

102. For this reason, the inversions of the given permutation are in a one-to-one correspondence with the inversions of the permutation of the indices of the elements for a normal arrangement of these elements, and hence the number of both kinds of inversions is the same.

103. Hint. In the permutation \( a_1, a_2, \ldots, a_n \), transfer the element \( a_1 \) to the first position, in the resulting permutation transfer \( a_2 \) to the second position, and so forth.

104. For example, 2, 3, 4, \ldots, \( n \), 1 or \( n, 1, 2, \ldots, n-1 \). Hint. In the proof, utilize the fact that one transposition can reduce the number of elements that stand in the permutation to the right (or to the left) of their places in the normal arrangement by not more than unity.

105. Hint. In the permutation \( a_1, a_2, \ldots, a_n \), carry the element \( b_1 \) to the first place by means of adjacent transpositions, then, via adjacent transpositions, carry element \( b_2 \) to the second place in the resulting permutation, and so on.

106. \( C_n^2 - k \). 107. \( \frac{2}{3} \) for \( C_n^2 \). Hint. Make use of the preceding problem.

108. Use adjacent transpositions to carry 1 to the first place, then 2 to the second place, and so on. Take into account that one adjacent transposition changes the number of inversions by unity.
148. Consider a series of permutations beginning with the
permutation 1, 2, ..., n obtained by the following series of transpo-
sitions: first carry unity to the last position, interchanging it with
each number on the right, then, in the same manner, transfer the
two to the next-to-last position, and so on, until we arrive at the
permutation n, n — 1, ..., 2, 1.

The assertion may also be proved by induction on the number k.

149. Solution. To derive the recurrence relation, note that if n
is a permutation with k inversions the number n + 1 is in the last
position, then all k inversions are formed by the numbers 1, 2, ..., n,
and there will be (n, k) such permutations; if n + 1 resides in the
next-to-last position, then it forms one inversion, and the numbers
1, ..., n form k — 1 inversions, and there will be (n, k — 1)
such permutations, and so forth; finally, if n + 1 lies in the
first position, then it forms n inversions (this is possible only if k = n),
and the numbers 1, 2, ..., n form k — n inversions, and there
will be (n, k — n) such permutations.

By arranging the numbers (n, k) in an array made up of rows
with the given n and of columns with the given k, we see, from the
recurrence relation, that each number of the (n + 1)th row is equal
to the sum of the numbers of the preceding row, reckoning from
the left of the number standing above the desired number (includ-
ing numbers equal to zero). If, for the sake of convenience of reckoning
the positions, we write out the zero values of (n, j) as well when
j > n, and if we take into account that (1, 0) = 1, (1, f) = 0 for
f = 1, then we obtain the following array of values (n, k):

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>6</td>
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<td>5</td>
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<td>9</td>
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<td>22</td>
<td>20</td>
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<td>9</td>
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<td>4</td>
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<tr>
<td>6</td>
<td>1</td>
<td>5</td>
<td>14</td>
<td>20</td>
<td>49</td>
<td>71</td>
<td>90</td>
<td>101</td>
<td>101</td>
<td>99</td>
<td>71</td>
<td>49</td>
<td>29</td>
<td>14</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

For example, the number of permutations of six elements with
seven or eight inversions is equal to 101.

150. Hint. In all permutations, replace the given arrangement
by the inverse.

151. (1 4 2) (3 5). The decrement is equal to 3. The substitution
is odd.

152. (1 2 3) (2 5) (4). The substitution is odd.

153. (1 4 2) (3 4 7) (5). The substitution is even.

154. (2 3 4 5) (6 7 8). The substitution is even.

155. (1 2 3 4) (5 6 7 8). The substitution is odd.

156. (1 2 3 4) (5 6 7 8). The substitution is odd.

157. (1 2 3 4) (5 6 7 8). The substitution is odd.

158. (1 2 3 4) (5 6 7 8). The substitution is odd.

159. (1 2 3 4) (5 6 7 8). The substitution is odd.

160. (1 2 3 4) (5 6 7 8). The substitution is odd.

161. (1 2 3 4) (5 6 7 8). The substitution is odd.

162. (1 2 3 4) (5 6 7 8). The substitution is odd.

163. (1 2 3 4) (5 6 7 8). The substitution is odd.

164. (1 2 3 4) (5 6 7 8). The substitution is odd.

165. (1 2 3 4) (5 6 7 8). The substitution is odd.

166. (1 2 3 4) (5 6 7 8). The substitution is odd.

167. (1 2 3 4) (5 6 7 8). The substitution is odd.

168. (1 2 3 4) (5 6 7 8). The substitution is odd.

169. (1 2 3 4) (5 6 7 8). The substitution is odd.

170. (1 2 3 4) (5 6 7 8). The substitution is odd.

171. (1 2 3 4) (5 6 7 8). The substitution is odd.

172. (1 2 3 4) (5 6 7 8). The substitution is odd.

173. (1 2 3 4) (5 6 7 8). The substitution is odd.

174. (1 2 3 4) (5 6 7 8). The substitution is odd.

175. (1 2 3 4) (5 6 7 8). The substitution is odd.


177. The identical substitution is even.

178. A. Hint. Arrange the numbers of the first row of the substitution
in increasing order and then pass from the identical substitution
to the given substitution via a series of transpositions in the
row.

180. Hint. To prove the existence of the decomposition into
transpositions of the number equal to the decrement, multiply the
substitution by the transposition of numbers that appear in one
of transpositions, note that when multiplying by one transposi-
tion the decrement cannot be increased by more than unity.
189. Hint. If \( P = P_1 P_2 \ldots P_n \) is any decomposition of the substitution \( P \) into transpositions, then make use of the equality

\[
P = \left( \begin{array}{ccc}
a_1 & a_2 & \ldots & a_n \\
a_{P_1} & a_{P_2} & \ldots & a_{P_n}
\end{array} \right).P_1P_2 \ldots P_n.
\]

184. Solution. If \( X \) is a substitution permutable with \( S \), then \( S = XX \). Whence \( X^{-1} XX = S \). Decompose \( S \) into cycles: \( S = (1, 2) (3, 4) \). \( X \) is again a cycle of length 2. \( S \) corresponds to them in the substitution \( X \). Thus the substitution \( X \) must carry all cycles of \( S \) into cycles of the same length, and since the decompositions of \( S \) into cycles is unique, the cycles either pass into themselves or come into another. Since each cycle of length two can be written in two ways, \( (1, 2) = (2, 1) \), \( (3, 4) = (4, 3) \), it follows that all substitutions that are permutable with \( S \) are

\[
\begin{align*}
(1234) & \quad (1234) \\
(1234) & \quad (1234) \\
(1234) & \quad (1234) \\
(1234) & \quad (1234)
\end{align*}
\]

The desired substitutions are

\[
\begin{align*}
1234 & \quad 1234 & \quad 1234 & \quad 1234 \\
1234 & \quad 1234 & \quad 1234 & \quad 1234 \\
1234 & \quad 1234 & \quad 1234 & \quad 1234 \\
1234 & \quad 1234 & \quad 1234 & \quad 1234
\end{align*}
\]

185. The desired substitutions are

\[
\begin{align*}
12345 & \quad 12345 & \quad 12345 & \quad 12345 \\
12345 & \quad 12345 & \quad 12345 & \quad 12345 \\
12345 & \quad 12345 & \quad 12345 & \quad 12345 \\
12345 & \quad 12345 & \quad 12345 & \quad 12345
\end{align*}
\]

186. Hint. Demonstrate that none of the numbers \( 1, 2, \ldots, m - 1 \) pass into zero and distinct numbers pass into distinct numbers.

187. \( 1, 2, 3, 4, 5, 6, 7, 8 \).

188. With the minus sign. 189. With the plus sign. 190. It is a term of a determinant. 191. With the minus sign. 192. It is a term of a determinant. 193. With the sign \((-1)^{n-1} \). 194. With the sign \((-1)^{n-1} \).

195. The sign \((-1) \).

196. With the sign \((-1)^n \).

197. \(-5, h = 1, i = 0, k = 2 \). 198. \( a_1 a_2 a_3 a_4 a_5 + a_3 a_4 a_5 a_1 a_2 \). 199. \( a_{10} a_{20} a_{30} a_{40} + a_{30} a_{40} a_{10} a_{20} \). 200. 192. 201. With the plus sign 202. \( n(1+1)=n(n-1) \).

193. \( a_1 a_2 a_3 \ldots a_m \). 204. \((-1)^n \cdot a_1 a_2 a_3 \ldots a_m \). Hint. That a polynomial of degree \( n \) cannot have more than \( n \) distinct roots. 208. \( x = 0, 1, 2, \ldots, n - 1 \). 209. \( a_0 + a_1 + \ldots + a_{n-1} \). 210. \( a_0 + a_1 + \ldots + a_{n-1} + \ldots + a_{n-1} \).

211. If \( n \) is even, the number of elements at even and odd sites is the same and is equal to \( \frac{1}{2} n^2 \). If \( n \) is odd, then the number of elements at even sites is equal to \( \frac{1}{2} (n^2 + 1) \); and at odd sites is equal to \( \frac{1}{2} (n^2 - 1) \).

212. The determinant is multiplied by \((-1)^{n-1} \). 213. The determinant is multiplied by \((-1)^{n-1} \). 214. The determinant remains unchanged. 215. The determinant does not change. Hint. Consider the general term of the determinant.

216. Hint. Transpose the determinant.

217. Hint. Transpose the determinant.

218. \( n = 4m \), where \( m \) is an integer. 219. \( n = 4m + 2 \), where \( m \) is an integer.

220. The determinant will vanish if it is of even order and will be doubled if it is of odd order. Hint. Decompose into a sum of determinants in terms of each column.

221. The determinant is multiplied by \((-1)^{n^2} \). 222. The determinant remains unchanged. Hint. Consider the sum of the indices of all elements that appear in the general term of the determinant.


224. The determinant must be unchanged.

225. The determinant will vanish if it is of even order and will be doubled if it is of odd order. Hint. Decompose into a sum of determinants in terms of each column.

226. The determinant is multiplied by \((-1)^{n^2} \). 227. The determinant is equal to zero. 228. The number of such determinants is equal to \( 1 \). Their sum is equal to zero.

229. \( 0, 236, 8a + 15b + 12c - 18d \). 230. \( 2a - 3b + c + 5d \).

231. \( a_{10} a_{20} a_{30} a_{40} \). 232. \( a_{10} a_{20} a_{30} a_{40} \). 233. \( a_{10} a_{20} a_{30} a_{40} \).

234. \( a_{10} a_{20} a_{30} a_{40} \). 235. \( a_{10} a_{20} a_{30} a_{40} \). 236. \( a_{10} a_{20} a_{30} a_{40} \). 237. \( a_{10} a_{20} a_{30} a_{40} \).

238. \( 0 \).

239. \( a_{10} a_{20} a_{30} a_{40} \).

240. \( a_{10} a_{20} a_{30} a_{40} \).

241. \( 0 \).

242. \( a_{10} a_{20} a_{30} a_{40} \).

243. \( 0 \).

244. \( a_{10} a_{20} a_{30} a_{40} \).

245. \( a_{10} a_{20} a_{30} a_{40} \).

246. \( a_{10} a_{20} a_{30} a_{40} \).

247. \( a_{10} a_{20} a_{30} a_{40} \).

248. \( a_{10} a_{20} a_{30} a_{40} \).

249. \( a_{10} a_{20} a_{30} a_{40} \).

250. \( a_{10} a_{20} a_{30} a_{40} \).

251. \( a_{10} a_{20} a_{30} a_{40} \).

252. \( a_{10} a_{20} a_{30} a_{40} \).

253. \( a_{10} a_{20} a_{30} a_{40} \).

254. \( a_{10} a_{20} a_{30} a_{40} \).

255. \( a_{10} a_{20} a_{30} a_{40} \).

256. \( a_{10} a_{20} a_{30} a_{40} \).

257. \( a_{10} a_{20} a_{30} a_{40} \).

258. \( a_{10} a_{20} a_{30} a_{40} \).

259. \( a_{10} a_{20} a_{30} a_{40} \).

260. \( a_{10} a_{20} a_{30} a_{40} \).
The text appears to be a series of mathematical expressions and proofs. It involves determinants, algebraic manipulations, and the application of various mathematical principles. The content is dense and technical, typical of advanced mathematics or a specialized field within mathematics.

For a more precise interpretation, each line would need to be translated into a coherent mathematical statement or proof. The expressions include the manipulation of determinants, algebraic equations, and the application of various mathematical theorems and properties.
350. \( \prod_{1 \leq i < k < n} (a_i b_k - a_k b_i) \). 351. \( \prod_{1 \leq i < k \leq n} \sin (a_i - a_k) \).

352. \( a_1 y_1 \cdots y_n \prod_{1 \leq i < k < n+1} (x_i y_k - x_k y_i) \), where \( a_i \) is the coefficient of \( x^i \) in the polynomial \( f_i (x, y) \).

353. \((-1)^{n-1} \prod_{i=1}^{n} x_i \prod_{n \geq i > k > 1} (x_i - x_k) \left[ \sum_{i=1}^{n} \frac{a_i}{x_i f_i (x_i)} \right], \) where \( |f(x)| = (x-x_i) (x-x_i^2) \cdots (x-x_n) \). Hint. Expand the determinant in terms of the first row.

354. \( (x_1 + x_2 + \cdots + x_n) \prod_{n \geq i > k > 1} (x_i - x_k) \). Hint. Compute the determinant

\[
\begin{vmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^n \\
1 & x_2 & x_2^2 & \cdots & x_2^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^n \\
1 & z & z^2 & \cdots & z^n \\
\end{vmatrix}
\]

in two ways: by expanding in terms of the last row and as a Vandermonde determinant. In both expressions, equate the coefficients of \( x^{n-1} \).

355. \( \prod_{n \geq i > k > 1} (x_i - x_k) \), where the sum is taken over all combinations of \( n-s \) numbers \( a_1, a_2, \ldots, a_{n-s} \) from the numbers 1, 2, \ldots, n.

356. \( \prod_{n \geq i > k > 1} (x_i - x_k) \), Hint. Express the ith element of the first column as \( 1 - \frac{a_i}{x_i} \) and represent the determinant as a difference of two determinants.

357. \( \sum_{i=1}^{n} (a_i + b_i) + \sum_{1 \leq i < k \leq n} (a_i - a_k) (b_k - b_i) \).

358. \((-1)^n \prod_{1 \leq i < k \leq n} (x_k y_1 - x_1 y_k) \). Hint. Apply the result of problem 355.

359. \( \prod_{n \geq i > k > 1} (a_i - b_i) + \prod_{1 \leq i < k \leq n} (a_i - a_k) (b_k - b_i) \).

360. \( (x_1^2 + \cdots + x_n^2) (a_1^2 + \cdots + a_n^2) \). 361. \( (a_3^2 - a_1^2) \).

362. \( \prod_{i=1}^{n} (a_i^2 b_{2i} - a_i b_{3n+1-i}) \).

363. \( \frac{n+1}{x^n} \).

364. \( n \) if \( n \) yields a remainder of 2 or 5 when divided by 6; 1 if \( n \) is divisible by 6 or yields a remainder of one; \( -1 \) if \( n \) yields a remainder of 3 or 4 when divided by 6. The answer can also be written down as

\[
D_n = C_{n+1} - 3C_{2n+1} + 3C_{2n+2} - 27C_{2n+3} + \cdots.
\]

Hint. Apply the method of recurrence relations discussed in the introduction to this section.

365. Hint. Obtain the recurrence relation \( D_n = D_{n-1} + D_{n-2} \).
306. If $n$ is odd, then $D_n = 0$; if $n$ is even, then $D_n = (\frac{-1}{2})^n$.

308. The first expression can be obtained directly by the method of recurrence relations, the second can be readily proved by the method of induction by using the relation $C^n_k + C^n_{k+1} = C^n_{k+1}$.

309. The integral part of the number $\frac{n}{2}$ denotes the integral part of the number $\frac{n}{2}$. Hint. Prove by induction on $n$ that $C_n^k$ is equal to the given determinant. And if $D_n$ is the determinant of the given problem, and $D_n^k$ is the determinant of problem 309 with $2\cos\alpha$ substituted for $a$, then $D_n = D_n^k - \cos n D_n^{k-1}$. The fact that the coefficients in the resulting expression of $\cos n\alpha$ in terms of $\cos \alpha$ are integral follows from the easily verifiable equality $\frac{n}{n-k} C^n_{n-k} = C^n_{n-k} - C^n_{n-k-1}$ and from the fact that all terms, except the last one, contain the factor 2, and the last term fails to involve $2\cos \alpha$ only for even $a$, but then $k = \frac{n}{2}$ and this term is equal to 2.

310. $\sin n\alpha = \sin \alpha (2\cos \alpha)^{n-2} - C^n_{n-4}(2\cos \alpha)^{n-3} + C^n_{n-5}(2\cos \alpha)^{n-4} - \ldots - C^n_{n-k}(2\cos \alpha)^{n-k} + \ldots$, where $\left[ \frac{n}{2} \right]$ denotes the integral part of the number $\frac{n}{2}$. Hint. Apply the method of recurrence relations.

312. From each row subtract the preceding row, and so forth until we arrive at a determinant of the same type as in the preceding problem.
upper left-hand corner as \(1 + 2\) and obtain the relation \(D_{n-1} = D_{n-2} + 2D_{n-3}\), where \(D_{n-1}\) is a determinant of the same type as in problem 379 but of order \(n - 1\).

386. \((x - 1)^n\). Hint. From each row subtract the preceding row and show that \(D_{n+1} = (x - 1)D_n\).

388. \(1 + 2 + 3 + \cdots + (n - 1) + (x - 1)^n\). Hint. Reduce this problem to the preceding problem.

390. \(x^n - \left(\frac{1}{2}\right)x^{n-1} + \left(\frac{n}{3}\right)x^{n-2} - \cdots + (-1)^n x\).

Hint. From each row subtract the preceding row, show that the method of mathematical induction.

391. \((-1)^n x^n + \frac{1}{2}x^{n-1} - \cdots - y^n\). Hint. In the lower right-hand corner set \(0 = x - x\), decompose into two determinants and either apply the method of recurrence relations or find \(D_n\) from the following two equations:

\[
D_n = -x D_{n-1} + x (-y)^{n-1}, \quad D_n = -y D_{n-1} + y (-x)^{n-1}.
\]

392. \(f(y) = (a - y)^n\).

393. \(f(y) = (a - y)^n\), where \(f(x) = (a - x)(a - 2x) \cdots (a - nx)\).

394. \(f(x) = f(y)\), where \(f(z) = (a_1 - z)(a_2 - z) \cdots (a_n - z)\).

395. \(a^n + b^n\). 396. \(a^{n+1} + b^{n+1} + \cdots + a + b\).

397. \(a^n + b^n + c^n + \cdots + a + b + c\).

398. \(\tan^n \frac{\alpha}{2} + \cot^n \frac{\alpha}{2}\).

399. \(n \prod_{i=1}^{p} (x_i + 2i + 1)\) or \((x_2 - 1^2) (x_3 - 2^2) \cdots (x_n - (n - 1)^2)\).

If \(n\) is even and \(x(z - 2^2) (z^2 - 4^2) \cdots (z^2 - (n - 1)^2)\) if \(n\) is odd. Hint. To each row add all the following rows, from each column subtract the preceding column and show that if \(D_n(x)\) is the given determinant, then \(D_n(z) = (x + n - 1)D_{n-1}(x - 1)\).

400. \(0\) if \(n > 2\), \(D_1 = a^p - z; D_2 = x a p (a^2 - 1) (1 - a)\).

401. \((-1)^n \left(\sum_{i=1}^{p} x_i^{2n-1} \right) a_{2n-1} - a_{2n-2}\).

402. \(a_1 a_2 \cdots a_n (a_{n+1} - 1) \cdots (a_n - 1)\).

403. \(abc_{1}c_2 \cdots c_n (c_{n+1} - 1) \cdots (c_1 - 1)\).

404. \(a (a + b) (a + 2b) \cdots (a + (n - 1) b) \left(\sum_{i=1}^{p} \frac{1}{a_i + b} \right)\).

405. \(\left[1 + \sum_{i=1}^{n} \frac{a_i}{x_i - 2a_i x_i} \right] \prod_{i=1}^{p} (x_i - 2a_i x_i)\). Hint. Use the result of problem 398.

406. \(\left[1 + \sum_{i=1}^{n} \frac{a_i^2}{x_i - 2a_i x_i} \right] \prod_{i=1}^{p} (x_i - 2a_i x_i)\).

407. \(1 - b_1 + b_1 b_2 - b_1 b_2 b_3 + \cdots + (-1)^n b_1 b_2 \cdots b_n\). Hint. Obtain the relation \(D_n = 1 - b_1 D_{n-1}\).

408. \((-1)^{n-1} b_1 a_n a_3 \cdots a_n + b_1 a_3 a_5 \cdots a_{n-1} - b_2 b_4 \cdots b_{n-1}\).

409. \((-1)^{n-2} x^{n-2}\). Hint. From each row subtract the next row.

410. \((-1)^n [x - 1^n - x^n]\). Hint. From each row subtract the preceding row, set \(1 = x + (1 - x)\) in the lower right-hand corner, and express the determinant as a sum of two determinants.

411. \(a_2 n + \sum_{i=1}^{n} (b_i - a_i)\). Hint. Multiply the second row by \(x^n\), the third by \(x^{n-1}\), and so on, and the \(n\)th by \(x\). Take \(x^n\) out of the first column, \(x^{n-1}\) out of the second column, \(x^{n-2}\) out of the third column, and so forth, and take \(x\) out of the \(n\)th column.

412. \(n! (1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n})\).

413. \(\frac{n(n+1)}{2} \left[1 + x \left(2 - \frac{4}{2n}\right)\right]\).

414. \(\frac{1}{n!} \left[1 + n (2 - \frac{4}{2n})\right] x\). 415. \(\left[\frac{1}{a} (\frac{n(n+1)}{2}) + \frac{x^{n+1}-1}{n+1} \right] + \frac{\alpha}{\beta}\).

416. \(\sum_{1 \leq i < k \leq n} (a_i - a_k) (b_i - b_k)^n\), where the product in the denominator is taken over all \(i, k\), which independently run through all values from 1 to \(n\). Hint. In each row, take outside the sign of the determinant the common denominator of the elements of that row. Show that the resulting determinant \(D'\) is divisible by all differences of the type \(a_i - a_k\) and \(b_i - b_k\) (\(i \neq k\)). Show that the quotient obtained from the division of \(D'\) by \(\prod_{1 \leq i < k \leq n} [(a_i - a_k) \times (b_i - b_k)]\) is a constant; to determine this constant, in \(D'\) put

\[
\frac{a_i}{b_i} = -b_i, \quad a_i = -b_i, \quad a_i = -b_i, \ldots, a_n = -b_n
\]

This can be done differently: from each row subtract the first row, and then from each column subtract the first column.
117. \[ \begin{bmatrix} 
(x_1 - x_3) (a_1 - a_3) \\
(x_2 - x_3) (a_2 - a_3) \\
(x_1 - x_2) (a_1 - a_2) 
\end{bmatrix} \]

Hint. Take advantage of the preceding problem.

118. \[ \frac{1}{(n + 1)! (n + 2)!} \]

Hint. Take advantage of the results of problem 416.

417. \( a_1 a_2 a_3 \ldots a_n \left( \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \right) \)

Hint. Use the recurrence relation \( D_n = \frac{1}{a_{n-1}} + a_{n-1} D_{n-1} = a_{n-1} D_{n-2} \) and apply the method of mathematical induction.

418. The continuant \( [a_1 a_2 a_3 \ldots a_n] \) is equal to the sum of all possible products of the elements \( a_1, a_2, \ldots, a_n \), one of which contains all these elements and the others are obtained from it by deleting one or several pairs of factors with adjacent numbers. Here, the term obtained by deleting all factors (for even \( n \)) is taken to be equal to 1;

\[ ([a_1 a_2 a_3 a_4]) = a_1 a_2 a_3 a_4 + a_1 a_2 a_4 + a_2 a_3 + a_4 + 1 \]

\[ ([a_1 a_2 a_3 a_4 a_5]) = a_1 a_2 a_3 a_4 a_5 + a_1 a_2 a_3 a_5 + a_2 a_3 a_4 + a_1 a_2 a_4 + a_1 a_3 a_5 + a_2 a_3 a_5 + a_3 a_4 + a_1 a_4 + a_2 a_5 + a_2 a_4 + a_3 a_5 + a_3 a_4 + a_4 + 1 \]

Hint. Verify the truth of this law for continuants of 1st and 2nd order and, assuming it to hold for continuants of \( (n - 1) \) and \( (n - 2) \), prove that it holds true for continuants of the \( n \)th order. To do this, derive the recurrence relation

\[ (a_1 a_2 \ldots a_n) = a_1 (a_2 a_3 \ldots a_n) + (a_1 a_2 \ldots a_{n-1}) \]

419. \( n \times n \times a_1 a_2 a_3 \ldots a_n \left( \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \right) \)

Hint. Use the recurrence relation \( D_n = \frac{1}{a_{n-1}} + a_{n-1} D_{n-1} = a_{n-1} D_{n-2} \) and apply the method of mathematical induction.

420. The continuant \( [a_1 a_2 a_3 \ldots a_n] \) is equal to the sum of all possible products of the elements \( a_1, a_2, \ldots, a_n \), one of which contains all these elements and the others are obtained from it by deleting one or several pairs of factors with adjacent numbers. Here, the term obtained by deleting all factors (for even \( n \)) is taken to be equal to 1;

\[ ([a_1 a_2 a_3 a_4 a_5]) = a_1 a_2 a_3 a_4 a_5 + a_1 a_2 a_3 a_5 + a_2 a_3 a_4 + a_1 a_2 a_4 + a_1 a_3 a_5 + a_2 a_3 a_5 + a_3 a_4 + a_1 a_4 + a_2 a_5 + a_2 a_4 + a_3 a_5 + a_3 a_4 + a_4 + 1 \]

Hint. Verify the truth of this law for continuants of 1st and 2nd order and, assuming it to hold for continuants of \( (n - 1) \) and \( (n - 2) \), prove that it holds true for continuants of the \( n \)th order. To do this, derive the recurrence relation

\[ (a_1 a_2 \ldots a_n) = a_1 (a_2 a_3 \ldots a_n) + (a_1 a_2 \ldots a_{n-1}) \]

421. \( (\sum a_i)^2 \)

424. \( 
\begin{align*}
(n + 1)! (n + 2)! \\
(2n + 1)!
\end{align*}
\)

Hint. Verify the truth of this law for continuants of 1st and 2nd order and, assuming it to hold for continuants of \( (n - 1) \) and \( (n - 2) \), prove that it holds true for continuants of the \( n \)th order. To do this, derive the recurrence relation

\[ (a_1 a_2 \ldots a_n) = a_1 (a_2 a_3 \ldots a_n) + (a_1 a_2 \ldots a_{n-1}) \]

425. \( (\sum a_i)^2 \)

426. \( 
\begin{align*}
(n + 1)! (n + 2)! \\
(2n + 1)!
\end{align*}
\)

Hint. Verify the truth of this law for continuants of 1st and 2nd order and, assuming it to hold for continuants of \( (n - 1) \) and \( (n - 2) \), prove that it holds true for continuants of the \( n \)th order. To do this, derive the recurrence relation

\[ (a_1 a_2 \ldots a_n) = a_1 (a_2 a_3 \ldots a_n) + (a_1 a_2 \ldots a_{n-1}) \]
Solution. The proof follows the same pattern as in the Laplace Theorem. We will show that any term of the product \( eM_1M_2 \ldots M_p \) is a term of the determinant \( D \). First let \( M_1 \) lie in the first \( k \) rows and the first \( k \) columns, \( M_2 \) in the next \( l \) rows and the next \( l \) columns, and so forth, and \( M_p \) in the last \( s \) rows and the last \( s \) columns. In that case the substitution (1) is an identical substitution and \( s = 1 \).

We now take the product of any terms of the minors \( M_1, M_2, \ldots, M_p \) in increasing order of the first indices of the elements. It contains one element each from every row and every column, and hence is the composition of elements, a term of \( D \). If in the second indices of the elements of the term of the minor \( M_1 \) there are \( a_1 \) inversions, then the sign of that product is \((-1)^{a_1} \ldots + a_p \). But the indices of the elements of two distinct minors \( M_1 \) and \( M_2 \) do not form inversions. Thus, \( a_1 + \ldots + a_p \) is the total number of inversions in the second indices of the elements of the product taken, and it will be a term of \( D \) in sign as well. Now let the minors \( M_1 \) be arranged in arbitrary fashion. Let us carry them into the above-considered position on the principal diagonal by the following permutations of rows and columns of \( D \). To begin with, carry the first row of the minor \( M_1 \), which has the number \( a_1 \), to the first position by interchanging it with all above-lying rows of \( D \). In doing so, we perform \( a_1 - 1 \) transpositions of rows, that is, just as many as there are inversions formed by the number \( a_1 \) in the upper row of the substitution (1) with the numbers that follow it. In the same manner, the row with the number \( a_2 \) is carried into the second position via the same number of transpositions of rows as there are inversions formed by \( a_2 \) in the next row of the substitution (1) with the numbers that follow it, and so on. The columns of \( D \) are interchanged in the same manner. If there are \( n \) inversions in the first row of the substitution (1) and \( n \) inversions in the second row, then \( e = (-1)^{a_1 + a_2} \), and altogether we perform \( a_1 + a_2 \) transpositions of rows and columns of \( D \). We therefore arrive at the new determinant \( D' \) for which

\[
D = eD'. \tag{2}
\]

By what has already been proved, any term of the product \( eM_1M_2 \ldots M_p \) will be a term of the determinant \( D' \) and by virtue of (2) any term of the product \( eM_1M_2 \ldots M_p \) will be a term of the determinant \( D \).

All terms of one and the same or of two distinct products \( eM_1M_2 \ldots M_p \) differ from one another in the composition of the elements and therefore are distinct terms of the determinant \( D \). It remains to prove that the total number of terms of all such products is equal to \( C_n^k \). If \( M_1 \) has already been chosen, then the minors \( M_2 \) can only lie in the remaining \( n-k \) rows and their number, for each choice of \( M_1 \), is equal to \( C_{n-k}^l \). For \( M_2 \) chosen, \( M_3 \) and \( M_4 \), the number of minors \( M_3 \) is equal to \( C_{n-k-l}^m \). So on; finally, for the chosen \( M_1, M_2, \ldots, M_{p-1} \), the number of products of the form \( eM_1M_2 \ldots M_p \) is

\[
C_n^k C_n^{k-1} C_{n-k-l}^m \ldots C_{n-k-l-m}^{m-1} \ldots C_{n-k-l-m}^{m-2} C_m^{m-1} \ldots C_1^1 \]

But the number of terms of the determinant \( D \) in each product \( eM_1M_2 \ldots M_p \) is equal to \( C_D(n-1) \). I.e., hence, the number of terms in all products \( eM_1M_2 \ldots M_p \) is equal to

\[
C_D(n-1) C_D(n-2) \ldots C_D(n-p) \cdot \frac{1}{p!} \]

476. We obtain by multiplying

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<tr>
<td>(-13) (-47) (-21)</td>
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<td>(-12) (-37) (-17)</td>
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</tbody>
</table>

The values of the given determinants are \(-5\) and \(19\), and the values of all determinants obtained are equal to \(-50\).

468. \((a^2 + b^2 + c^2 + d^2) \sin^2 \left( a - b \right) \sin \left( c - d \right) \sin \left( b - c \right) \sin \left( a - d \right) \sin \left( a - c \right) \sin \left( b - d \right)\).

470. \(0 \) for \( n > 2 \); \( (x_1 - x_2) (y_1 - y_2) \) for \( n = 2 \). Hint. Express

\[
\begin{pmatrix}
1 & x_1 & 0 & \ldots & 0 \\
1 & y_1 & 0 & \ldots & 0 \\
1 & x_2 & 0 & \ldots & 0 \\
1 & y_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & 0 & \ldots & 0 \\
1 & y_n & 0 & \ldots & 0
\end{pmatrix}
\]

as a product of determinants.

471. \(0 \) if \( n > 2 \), \( (x_1 - x_2) \sin \left( a_1 - a_2 \right) \sin \left( b_1 - b_2 \right) \) if \( n = 2 \).

472. \(0 \) if \( n > 2 \), \( \sin^2 \left( a_1 - a_2 \right) \) if \( n = 2 \).

473. \(0 \) if \( n > 2 \), \( -\sin^2 \left( a_1 - a_2 \right) \) if \( n = 2 \).

474. \(\left| \begin{array}{cc}
(a_1 - a_2)(b_1 - b_2)
\end{array} \right| \)

475. \(\left| \begin{array}{cc}
(a_1 - a_2)(b_1 - b_2)
\end{array} \right| \)
476. $(n-1)^2$ $[(n-1)]^n$. Hint. Write the element in the $i$th row and the $k$th column as $[(k-1)^n-1]$ and expand via the formula of the power of a binomial or directly take advantage of the result of the preceding problem.

477. $\prod_{i=1}^{n} (x_i-x_j)^3$. 478. $\prod_{i=1}^{n} (x_i-x_j)$ $\prod_{i=1}^{n} (x_i-x_j)^2$. Hint. Express the following as a product of determinants:

$$
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{vmatrix}
$$

The product is set up in terms of rows.

479. Hint. Multiply the given determinant by the Vandermonde determinant

$$
v = \begin{vmatrix}
1 & e_1 & \cdots & e_{n-1} \\
e_1 & e_1^2 & \cdots & e_1^{n-1} \\
ev & e_2 & \cdots & e_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & e_n & \cdots & e_n^{n-1}
\end{vmatrix}
$$

481. $(1 - x^n - 1)$. Hint. Make use of the result of problem 479 and the equation $(1 - x^n - 1) = 1 - x^n$, where $e_1, e_2, \ldots, e_n$ are $n$th roots of unity. It is simpler to compute this determinant as a special case of the determinants of problem 325.

483. $(a+b+c+d)(a-b+c-d)(a+b-c+d)(a-b-c+d) = (a^2 - b^2 + c^2 - d^2)$

484. $1+(-1)^n n^2 = 0$ for odd $n$.

485. $(-1)^n \frac{(n+1)(n+2)}{2n}$ for even $n$.

486. Hint. Compute the first determinant by using the result of problem 479.

487. $(n-2p)(n-2p)$ if $n$ and $p$ are relatively prime; 0 if $n$ and $p$ are not relatively prime. Hint. Use the result of problem 479 and the properties of the $n$th roots of unity. In particular the fact that when $p$ is relatively prime to $n$, the numbers $e_1, e_2, \ldots, e_p$ are again all values of $\frac{1}{n}$ and for $p$ not relatively prime to $n$ there is an $e_p = \pm$ for which $e_p^2 = 1$.

488. $(a+b)^n - p(a-b)^n - p(a-b)^n$ if $n$ and $p$ are relatively prime; 0 if $n$ and $p$ are not relatively prime. Hint. Make use of the hint of the preceding problem.

489. $2n^2 \left( \cos \frac{n\pi}{n} - 1 \right)$. Hint. Set $\cos \frac{n\pi}{n} = \frac{1}{2} - \frac{1}{2}$, where $e = \cos \frac{n\pi}{n} + i \sin \frac{n\pi}{n}$, and make use of the result of problem 479 and also of the fact that for any $a$ we have $\frac{1}{n} \sum_{i=1}^{n} (a - \omega_{n,i})^m = 0$ and $e^{\frac{n\pi}{n}} = 1$.

490. $\frac{\cos \theta - \cos (\theta + 1)}{2(\cos \theta)} = 2n^3 \sin^2 \frac{\theta}{2} \left[ \cos \left( \frac{a + \frac{n}{2}}{2} - \cos \left( \frac{a + \frac{n+2}{2}}{2} \right) \right) \right]$

491. $\left( a \right) \left( n+1 \right) (2n+1) n^2 = \frac{2n-1}{12} (n+1)(2n+1)$

492. $(1-\alpha)^n (\alpha+1) (2n+1) n^2 = \frac{2n-1}{12}$

493. $(\alpha_1)(\alpha_2) \ldots (\alpha_n)$, where $(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1}$, and $\eta_1, \eta_2, \ldots, \eta_n$ are all values of the root $\beta = \eta$. For example, $\eta_1 = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$. Hint. Multiply the given determinant by the Vandermonde determinant made up of the numbers $\eta_1, \eta_2, \ldots, \eta_n$.

494. $(\alpha_1)(\alpha_2) \ldots (\alpha_n)$, where $(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all values of the $n$th root of $\beta$.

495. Hint. Denoting the $n$th roots of $\beta$ by $\eta_k = \cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n}$ for $k = 0, 1, 2, \ldots, 2n-1$, show that the numbers $\sin \frac{k\pi}{n}$ yield all the $n$th roots of unity and $a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3 + \ldots + a_n \eta_n$ = $(a_1 + a_{n+1}) + (a_2 + a_{n+2}) \eta_1 + (a_3 + a_{n+3}) \eta_2 + \ldots + (a_n + a_{2n}) \eta_n$ and the number $\eta_k$ with odd indices $k$ yield all the $n$th roots of $-1$. We have $a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3 + \ldots + a_n \eta_n = (a_1 + a_{n+1}) + (a_2 + a_{n+2}) \eta_1 + (a_3 + a_{n+3}) \eta_2 + \ldots + (a_n + a_{2n}) \eta_n$.

496. The product of two numbers, each of which is the sum of the squares of four integers, is itself the sum of the squares of four integers. Hint: Square each of the determinants.
497. The product of two numbers, each of which is equal to the value of the form \( x^2 + y^2 + z^2 - 3xyz \) for integral values of \( x, y, z \), is itself a number of the same type. Hint. Compute the product of the determinants

\[
\begin{vmatrix}
 a & b & c \\
 c & a & b \\
 b & c & a
\end{vmatrix} \quad \begin{vmatrix}
 a' & b' & c' \\
 c' & a' & b' \\
 b' & c' & a'
\end{vmatrix}
\]

by multiplying the rows of the first by the columns of the second.

498. Hint. In the product of the determinants

\[
\begin{vmatrix}
 a & 1 & b \\
 b & 1 & a \\
 c & 1 & b
\end{vmatrix} \quad \begin{vmatrix}
 a' & 1 & b' \\
 b' & 1 & a' \\
 c' & 1 & b'
\end{vmatrix}
\]

made up by multiplying columns by columns, multiply the third column by \( \alpha = a' - b' + c' \) and take the factor \( \alpha \) out of the second row of the determinant. Then subtract the first and second columns from the third column.

499. Hint. Write down the determinant \( D \) in the form

\[
D = \sum_{k=1}^{n} b_1, b_2, ..., b_m, h_m, a_{j_1}, a_{j_2}, ..., a_{j_m}
\]

where all sums of the \( j \)th column are taken over one and the same index \( k_j = 1, 2, ..., n \); then expand \( D \) into a sum of \( n^m \) determinants with respect to columns; in each summand of the \( j \)th column take \( b_{j_k} \) outside the sign of the determinant and show that

\[
D = \sum_{k=1}^{n} b_1, b_2, ..., b_m, a_{j_1}, a_{j_2}, ..., a_{j_m}
\]

where the summation indices vary from one to \( n \) independently.

Note that \( A_{j_1}, A_{j_2}, ..., A_{j_m} = 0 \) if there are equal indices among the indices \( k_1, k_2, ..., k_m \). From this derive the statement (2) and for \( m = n \) prove that for any indices \( i_1, i_2, ..., i_m \), where \( 1 \leq i_1 < i_2 < ... < i_m \leq n \), all summands of the sum (3) in which the indices \( k_1, k_2, ..., k_m \) form any permutations of the numbers \( i_1, i_2, ..., i_m \) have a sum equal to \( A_{i_1}, A_{i_2}, ..., A_{i_m} \). Hence this obtain statement (1).

500. Hint. Complete the matrices \( A \) and \( B \) to obtain square matrices \( A \) and \( B \) of \( m - n \) columns consisting solely of zeros.

501. Hint. Apply the theorem of problem 499 to the matrices

\[
\begin{pmatrix}
 a_1 & a_2 & ... & a_n \\
 b_1 & b_2 & ... & b_n
\end{pmatrix} \quad \begin{pmatrix}
 c_1 & c_2 & ... & c_n \\
 d_1 & d_2 & ... & d_n
\end{pmatrix}
\]

503. Hint. Take advantage of the identity of the preceding problem.

504. Hint. Apply the theorem of problem 499 to the matrices

\[
\begin{pmatrix}
 a_1 & a_2 & ... & a_n \\
 b_1 & b_2 & ... & b_n
\end{pmatrix} \quad \begin{pmatrix}
 \tilde{a}_1 & \tilde{a}_2 & ... & \tilde{a}_n \\
 \tilde{b}_1 & \tilde{b}_2 & ... & \tilde{b}_n
\end{pmatrix}
\]

505. Hint. Take advantage of the identity of the preceding problem.

506. Hint. By multiplying \( D \) and \( D' \) with respect to rows, show that \( D'\delta = D\delta \), whence follows (1) for \( D = 0 \). When \( D = 0 \), consider the case where all elements of \( D \) are zero. If \( D = 0 \), but at least one element \( a_{i,j} \neq 0 \), then add to the \( i \)th row of \( D \) multiplied by \( a_{i,j} \), the second multiplied by \( a_{i,j} \), the second by \( a_{i,j} \), and show that \( a_{i,j}D' = 0 \).

The case of \( D = 0 \) may be omitted if we assume the elements of \( D \) to be independent variables instead of numbers. Then the determinant will be a polynomial different from zero, and we prove that (1) is an identity; this means it is true for all numerical values of the variables \( a_{i,j} \), irrespective of whether \( D \) vanishes or not.

507. Hint. First consider the case where \( M \) lies in the upper left-hand corner. Multiplying with respect to rows of \( D \) the minor \( M \) written in the form

\[
\begin{pmatrix}
 A_11 & ... & A_m, m+1 & ... & A_1n \\
 A_m, m+1 & ... & A_1n \\
 0 & ... & 0 & 1 & 0 \\
 0 & ... & 0 & 0 & 1
\end{pmatrix}
\]

show that \( DM' = D' \delta M \) and \( M' = D' \delta M \) (the case \( D = 0 \) may be omitted as indicated in the preceding problem, that is, by regarding \( D \) as a polynomial in \( n^m \) unknowns \( a_{i,j} \)). Then reduce the general arrangement of \( M \) to the one under consideration by permutations of rows and columns; for this purpose show that when two adjacent rows (or columns) are permuted, there occurs a similar permutation of rows (or columns) in the adjugate determinant \( D' \) and, besides, all elements of \( D' \) change sign.

508. Hint. Use the preceding problem.


510. Hint. Apply the equation of problem 507 for \( m = n - 1 \).

511. Hint. Apply the equation of problem 507 with \( n = m \) substituted for \( m \).

512. Hint. Use the value of the adjugate determinant \( D' \) to find the value of the determinant \( D \) and apply the equality of problem 510. Show that the problem has \( n - 1 \) solutions.

513. Hint. Express the first determinant as the square of the Vandermonde determinant made up of the numbers \( 0, x_1, x_2, ..., x_n \).

514. Hint. Use the value of the adjugate determinant \( D' \) to find the value of the determinant \( D \) and apply the equality of problem 510. Show that the problem has \( n - 1 \) solutions.
515. Hint. Consider the products $DA$ and $AD$, where $D$ is the given determinant and $A$ is a determinant of the same order as $D$ obtained by interchanging the $i$th and $j$th rows from a determinant having ones in the principal diagonal and zeros elsewhere.

516. Hint. Consider the products $DA$ and $AD$, where $D$ is the given determinant and in $A$ the elements of the principal diagonal are equal to $1$, the element in the $i$th row and the $j$th column is equal to $a$, and all other elements are zero.

517. Hint. Set $\det A = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n$.


520. The determinant is equal to twice the area of the triangle $M_1M_2M_3$ if the direction of the smallest rotation of the ray $M_1M_2$ to coincidence with $M_3M_4$ coincides with the direction of the smallest rotation from the positive direction of $Ox$ to the positive direction of $Oy$, otherwise it is equal to twice the area of the triangle $M_1M_2M_3$ with the minus sign.

Section. The indicated transformations of the coordinates are given by the formulas:

$$x = x' \cos \alpha - y' \sin \alpha + x_0, \quad y = x' \sin \alpha + y' \cos \alpha + y_0.$$ 

From this, multiplying with respect to rows, we obtain:

$$\begin{vmatrix}
 x' & y' & 1 \\
 x & y & 1 \\
 1 & 1 & 1 \\
\end{vmatrix} = \begin{vmatrix}
 \cos \alpha & -\sin \alpha & x_0 \\
 \sin \alpha & \cos \alpha & y_0 \\
 0 & 0 & 1 \\
\end{vmatrix} = \begin{vmatrix}
 x_0 & y_0 & 1 \\
 x & y & 1 \\
 1 & 1 & 1 \\
\end{vmatrix}.$$

But the second determinant in the left-hand member of the equation is equal to $1$. This proves the invariability of the given determinant under the indicated transformations. Now carry the origin of coordinates to the point $M_3$ and rotate the axes so that the new axis of abscissas goes along $M_4M_5$. The new coordinates of the points $M_1, M_2, M_3$ will be $x_i = M_3M_i, \quad y_i = \pm h$, where $h$ is the altitude of the triangle $M_1M_2M_3$. The choice of the plus or minus sign is connected with the orientation of the triangle by the rule given above: $y_1 = y_2 = y_3 = 0$. For that reason the determinant takes the form:

$$\begin{vmatrix}
 x_1 & y_1 & 1 \\
 x_2 & y_2 & 1 \\
 x_3 & y_3 & 1 \\
 0 & 0 & 1 \\
\end{vmatrix} = \pm M_3M_1, h = \pm 2a.$$

where $S$ is the area of the triangle $M_1M_2M_3$.

521. The determinant is equal to the area of a parallelogram constructed on line segments joining the origin to the points $M_1, M_2, M_3, M_4$ the area being taken with the plus sign if the directions of the shortest rotation from $OM_1$ to $OM_2$ and from $OM_2$ to $OM_3$ coincide, and with the minus sign if these directions are opposite. The determinant does not change under a rotation of the axes but may change under a translation of the origin. Hint. Apply the result of the preceding problem by taking the coordinate origin for the third point.

522. $R = \frac{abc}{4x}$. Hint. Take the centre of a circumscribed circle for the coordinate origin and make use of the relations:

$$R^2 - (x-x_j)(x-x_j) + (y-y_j)^2 = \frac{1}{2}[(x_1-x_j)^2 + (y_1-y_j)^2] \quad (i, j = 1, 2, 3).$$

and also the result of problem 520.

523. Hint. Show that the square of the determinant is equal to 1. To determine the sign, make $\sum a_i^2$ of the continuity of the determinant in the totality of all variables $a_1, a_2, a_3$.

524. The determinant is equal to the volume of a parallelepiped constructed on line segments joining the coordinate origin $O$ to the points $M_1, M_2, M_3, M_4$ or is equal to six volumes of the tetrahedrons $OM_1M_2M_3, Oxyz$ taken with the plus sign if the orientations of the triangles $OM_1M_2M_3$ and $Oxyz$ are the same, and with the minus sign if these orientations are opposite (orientations are assumed to be the same if after coincidence—via rotation of the tetrahedron $Oxyz$—of the axis $Ox$ with $OM_1$ and of the plane $Oy$ with $OM_2, OOM_3$ if $Oy$ and $OM_3$ lie to one side of $Ox$, the rays $Ox$ and $OM_3$ lie to one side of the plane $Oy$; orientations are assumed to be opposite if the rays are on different sides). Hint. If $a_1, a_2, a_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$, respectively, the cosines of the new axes $Ox', Oy', Oz'$, with the old axes, then the old and new coordinates are connected by the relations:

$$x = a_1x' + \beta_1y' + \gamma_1z', \quad y = a_2x' + \beta_2y' + \gamma_2z', \quad z = a_3x' + \beta_3y' + \gamma_3z'.$$

Use this fact and the results of the preceding problem to prove the invariability of the determinant of the present problem. To determine the geometric meaning of the determinant, rotate the system of coordinates $Oxyz$ as indicated above for determining the like and opposite orientations of the tetrahedrons.

525. $V = abc \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma}$. Hint. Compute $V^2$ making use of the result of the preceding problem.

526. Hint. On the rays $OA, OB, OC$, take the points $M_1, M_2, M_3, M_4$ at a distance of 1 from the origin and apply the result of problem 524.

527. The determinant is equal to six volumes of a tetrahedron with vertices $M_1, M_2, M_3, M_4$ taken with the plus sign if the tetrahedron of rays from $M_4$ to each of the points $M_1, M_2, M_3, M_4$ has the same orientation as the tetrahedron $Oxyz$, and with the minus sign otherwise. Hint. Translate the origin to the point $M_4$ and apply the result of problem 524. An alternative way is to proceed as in the solution of problem 520, making use of problem 523. Then problem 524 will turn out to be a special case of the given problem (like what it was on the plane in problem 521).

528. $R = \frac{1}{24}V \times \frac{x}{V} = \frac{1}{24}R \times \frac{x}{V}$.

$V$ is the volume of the tetrahedron and $l_j = l_j/i = 1, 2, 3, 4$ is the length of the edge joining the vertices $(x_j, y_j, z_j)$ and $(x_j, y_j, z_j)$. In the case of a regular tetrahedron with edge $a$ we obtain $R = \frac{a}{4}$. 

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Apply the result of the preceding problem and the relation
\[ R^2 = z_i z_j - b u_j - z_i z_j = \frac{1}{2} \left[ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \right], \]
which is true on the assumption that the origin has been translated to the centre of a circumscribed sphere.

539. Hint. When proving statement (2), show that the vector
\[ a_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \]
may be represented as \( a_i = a_{i1} + a_{i2} + \ldots + a_{in} \). Furthermore, show that the function \( f(a_1, a_2, \ldots, a_n) \) changes sign when two vectors are interchanged. For example, when proving this with respect to the vectors \( a_1 \) and \( a_2 \), consider \( f(a_1 + a_2, a_3, a_4, \ldots, a_n) \).

539. Hint. Prove that the function \( f(a_1, a_2, \ldots, a_n) = |A B| \) of the rows of matrix \( A \) has the properties (a) and (b) and that
\[ f(c_1, c_2, \ldots, c_n) = |B|. \]
Set \( f(c_1, c_2, \ldots, c_n) = 1 \) for arbitrary \( i_1, i_2, \ldots, i_n = 1, 2, \ldots, n \) (identical or distinct). By virtue of (a), assuming
\[ a_i = \sum_{j=1}^{n} a_{ij}, \]
we get
\[ f(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_{i1}, a_{i2}, \ldots, a_{in}. \]
This defines the function \( f(a_1, a_2, \ldots, a_n) \). Clearly, it does not change under an interchange of vectors, that is, in the case of a field of characteristic 2 it changes sign. For that reason, (b') is fulfilled and (b) is obviously not fulfilled.

532. \((-1)^{n+1} C_{n-1}^2 = \frac{n^2}{2} - n. \) Solution. Multiplying the given determinant \( D \) by itself and noting that \( e^{k-1} \) if and only if \( k \) is divisible by \( n \), we obtain
\[
D^2 = \begin{vmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 \\
\end{vmatrix}
\]
and
\[ D = (-1)^{n+1} C_{n-1}^2 \text{ } n^2, \]
for the modulus of \( D \) we get \( |D| = \frac{n^2}{2} \). It is known that the determinant of \( D \) is equal to the product of the numbers 1, \( e^{-1/2}, \ldots, e^{-1/2} \) and then setting
\[ e^k = \cos \frac{
}{n} + i \sin \frac{
}{n}, \]
we obtain
\[ \prod_{0 \leq j < n \leq 1} (e^k - e^j) = \prod_{0 \leq j < n \leq 1} (e^{k-j} - e^{k+j}) \]
For the values of \( j \) and \( k \) considered above it is always true that
\[ 0 < k - j < n \text{ and, hence, } \sin \frac{(k-j)}{n} \geq 0. \]
542. Solution. If all $A_{1j} = 0$, then we can put
$$A_j = B_j = 0 \quad (i = 1, 2, \ldots, n).$$

For example, suppose the cofactors of the elements of the last column
are all zero (the reasoning is similar in the case of a different
column). Since $i = 0$, the cofactors of the elements of two columns
are proportional (see problem 593):
$$\frac{A_{1j}}{A_{1n}} = \frac{A_{2j}}{A_{2n}} = \cdots = \frac{A_{nj}}{A_{nn}} = \frac{B_j}{C_j} \quad (j = 1, 2, \ldots, n - 1),$$

where the fraction $\frac{B_j}{C_j}$ will be regarded as irreducible, from this we have
$$A_{ij} = A_{in} B_j \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n - 1). \quad (4)$$

But $A_{ij}$ is a polynomial in $x_1, x_2, \ldots, x_n$ and the fraction $\frac{B_j}{C_j}$ is
irreducible. Therefore, $A_{ij}$ must be divisible by $C_j$ ($i = 1, 2, \ldots, n - 1$) and, hence, by the least common multiple of all
$C_j$ ($j = 1, 2, \ldots, n - 1$). Denoting this least multiple by $B_n$, we obtain
$$A_{in} = A_j B_n \quad (i = 1, 2, \ldots, n). \quad (2)$$

where all $A_j$ are polynomials in $x_1, x_2, \ldots, x_n$. Set $B_j = B_{j1}, B_{j2}$
($i = 1, 2, \ldots, n - 1$). All $B_j$ are polynomials in $x_1, x_2, \ldots, x_n$ and from (4) we find
$$A_{ij} = A_{in} B_j \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n - 1). \quad (3)$$

The equations (2) and (3) prove the theorem, in particular, for the
determinant $A$ we can put $A_1 = \alpha_1, A_2 = \beta_2, A_3 = -\beta_3, A_4 = B_3 = \gamma = 0$.

543. Solution. Let $D_{2n}$ be a skew-symmetric determinant of order $2n$. Carry out induction on $n$. For $n = 1$ the theorem holds because
$D_2 = \alpha_1$. Suppose the theorem holds true for the number $n$ and then prove it for $n + 1$. Crossing out of the given determinant $D_{2n+2}$
the last row and the last column, we get the skew-symmetric deta?
iminant $D_{2n+1}$ equal to zero. Its elements may be regarded as polynom-
ial elements lying above the principal diagonal with integer
coefficients. By the theorem of the preceding problem, the cofactors
of the elements of $D_{2n+2}$ are of the form $A_{ij} = A_{in} B_j$ ($i = 1, 2, \ldots, 2n + 1$), where $A_j$ and $B_j$ are polynomials in the same unknowns.

By transposing and changing the signs of the elements, and since they are of even order $2n$, it follows that $A_{ij} = A_{ji}$. Or $A_{1j} B_j = A_j B_j$.

In the left we have a polynomial. But the square of an irreducible fraction
cannot be a polynomial. Hence, $\mu A_j = c_j$ is a polynomial. Apply
the expansion given in problem 341, we find
$$D_{2n+3} = \left( \sum_{i=1}^{2n+1} A_{ii} a_i \right) \left( \sum_{i=1}^{2n+1} B_j a_i \right) = \sum_{i=1}^{2n+1} A_{ii} B_j a_i \quad (5)$$

This proves the assertion of the problem. The proof does not yield a
convenient technique for actually computing the polynomial whose
square is equal to the given skew-symmetric determinant of even
order. Such rules are given in problems 545 and 546.

544. Hint. Consider two terms in which the substitution
of the indices has the cycle $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ ($n$ is odd and greater than unity)
and in the other the cycle $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Consider separately
the case $h = 1$.

545. Hint. When proving (4) on the basis of the given pair $N_j, \gamma_j$ of reduced Pfaffian products, restore the notation (1) of the substi-
tution of the desired term. Bearing in mind that if this term is written a
$$\pm \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \beta_1, \beta_2, \beta_3, \ldots, \beta_n, \gamma_1,$$
then $N_j$ consists of elements occupying odd sites in the product, and $N_j'$
consists of elements occupying even sites. For example, in $N_j$ we take
the element with the first index $\alpha_1 = 1$. The second index $\alpha_2$
yields the second element of the first cycle. In $N'_j$ we take the element
one of the indices of which is $\alpha_2$. If the other index is $\alpha_3$, the cycle
closes; if it is $\alpha_3$, then this is the third term of the cycle, and so on.

Show that all cycles in the resulting substitution are of even length.
In the proof (2) note that $N_j = N'_j$ and $N_j = N_j'$. Determine the
sign of the term as $(-1)^s$, where $s$ is the number of cycles of the ap-
propriate substitution. Derive assertion (3) from (1) and (2) and from
the theorem of the preceding problem.

546. Hint. Show that each term of the Pfaffian $p_n$ involves
and one only element of the $n$th column of $D_n$ and in each term of $p_n$ show
that the only thing left in the brackets will be $\rho_{2n}$ with the sign
$$(-1)^{n-1} = (-1)^{n-1}.$$
549. Hint. Expand, via the formula of problem 541, the determinant

\[
D' = \begin{vmatrix}
    a_{11} & \cdots & a_{1n} & x_1 \\
    \vdots & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \vdots & \vdots \\
    -a_{1n} & \cdots & a_{nn} & x_n \\
    -x_1 & \cdots & -x_n & 0
\end{vmatrix}
\]

obtained by bordering \( D \) and equate it to the square of the Pfaffian for \( D' \) by applying the formula of problem 546. In the resulting equation, put \( x_1 = x_j = 1, \ x_k = 0 \) for \( i \neq k \neq j \).

550. Hint. Using the fact that the product of two nonzero polynomials is nonzero, show that if \( D = AB \) is the presumed factorization and some term of the polynomial \( A \) contains \( a_i \), then no term of \( B \) contains elements of the first row (or column). From this fact derive that no matter what the \( i, j = 1, 2, \ldots, n \), there is a term in \( A \) containing \( a_i \) but no term in \( B \) contains \( a_j \).

551. Hint. When proving (2), determine \( \Delta_{n-k} \) by proceeding from the numbering \( t_1, t_2, \ldots, t_n \) of the combinations of the \( n \) numbers \( 1, 2, \ldots, n \) taken \( n-k \) at a time, which numbering is connected with the numbering \( s_1, s_2, \ldots, s_n \) that determines \( \Delta_k \) so that \( t \) contains those \( n-k \) numbers that do not appear in \( s_i \). If \( t_i \) is the sum of the numbers from the combination \( s_i \), then take the factor \((-1)^{i-1} \) out of the \( i \)th row and the \( i \)th column \( (i=1, 2, \ldots, n) \) of the determinant \( \Delta_{n-k} \). When proving (4), show, using the equation of item (3) and the irreducibility of \( D \) established in the preceding problem and also the degree of \( D \) and \( \Delta_k \) relative to the elements \( a_{ij} \), that \( \Delta_k = cD^{n-1}_{k-1} \), where \( c \) is independent of the elements \( a_{ij} \). To determine \( c \), show that both \( \Delta_k \) and \( D^{n-1}_{k-1} \) contain the term \( (a_{11}a_{22} \cdots a_{nn})^{n-k} \) with coefficient unity.

552. Hint. Show that \( P_n = Q_n = 1 \). Hint. Show that \( Q_n = P_\beta \).

553. Hint. Show that \( d_{ij} = \sum_{h=1}^{n} p_{hi} p_{kj} f^j(k) \), where \( p_{ij} \) are the same in the preceding problem.

### Chapter II. Systems of Linear Equations

568. \( x = 2, \ y = -1, \ z = \frac{3}{2}, \ t = 0. \)

569. \( x = -3, \ y = 0, \ z = -\frac{1}{2}, \ t = \frac{3}{2}. \)

570. \( x = -2, \ y = -3, \ z = -\frac{3}{2}, \ t = \frac{1}{2}. \)

582. The system does not have a solution. 583. The system does not have a solution.

584. Changing the numbering of the unknowns does not bring the system to an equivalent one, but when solving the system of equations it is permissible, provided that after the solution is complete we revert to the original numbering. Hint. Show that after the transformations of type (a), (b), (c), any equation of the new system can be expressed linearly in terms of an equation of the old system, and conversely.

585. \( x_1 = -1, \ x_2 = 3, \ x_3 = -2, \ x_4 = 7. \)

586. \( x_1 = 2, \ x_2 = 1, \ x_3 = -3, \ x_4 = 4. \)

587. \( x_1 = -2, \ x_2 = 1, \ x_3 = 4, \ x_4 = 3. \)

588. \( x_1 = 0, \ x_2 = -2, \ x_3 = \frac{1}{2}, \ x_4 = -3. \)

589. \( x_1 = 1, \ x_2 = -\frac{1}{2}, \ x_3 = 2, \ x_4 = -3. \)

590. \( x_1 = \frac{1}{4}, \ x_2 = -\frac{1}{3}, \ x_3 = 2, \ x_4 = -3. \)

591. \( x_1 = 105, \ x_2 = 7, \ x_3 = -10, \ x_4 = 1. \)

592. \( x_1 = 5, \ x_2 = 2, \ x_3 = 3, \ x_4 = 2, \ x_5 = 1. \)

593. \( x_1 = 3, \ x_2 = -5, \ x_3 = 4, \ x_4 = -2, \ x_5 = 1. \)

594. \( x_1 = \frac{1}{4}, \ x_2 = 2, \ x_3 = -2, \ x_4 = 2, \ x_5 = -\frac{1}{4}. \)

### Chapter II. Systems of Linear Equations

595. The system is indeterminate, that is, it has an infinity of solutions; \( x_1 \) and \( x_2 \) can be expressed in terms of \( x_3 \) and \( x_4 \) thus: \( x_1 = -\frac{1}{5} - 20x_3 + 17x_4, \ x_4 = -\frac{1}{7}x_3 - 5x_4 \); note that \( x_3 \) and \( x_4 \) can assume any values.

596. The system is indeterminate. The general solution is \( x_1 = \frac{1}{10} (6 - 15x_2 - x_4), \ x_2 = \frac{1}{5} (1 + 4x_4), \) where \( x_2 \) and \( x_4 \) assume arbitrary values.

597. The system is inconsistent, which means it does not have any solutions.

598. The system has no solutions.

599. If \( a_1, a_2, \ldots, a_\eta \) are all elements of a field, then the polynomial \( f(z) = (z - a_1)(z - a_2) \cdots (z - a_\eta) \) is equal to zero as a function, but \( x^n \) has a coefficient of unity.

580. \( f(z) = x^2 - 5x + 3 \). 580. \( f(z) = z^2 - 5z^2 + 7. \)

581. For a given asymptotic direction, it is possible to draw a parabola (and only one) not above the \( n \)th degree through any \( n + 1 \) points.
distinct points in a plane, of which no two points lie on the straight line of the asymptotic direction.

598. \( z = \frac{x}{3} - \frac{y}{3} \). 599. \( x = \frac{1}{\alpha} (a + b + c + d) \), \( y = \frac{1}{\alpha} (a - b - c + d) \).

592. \( x = \frac{1}{\alpha} (a^2 + b^2 + c^2 + d^2) \), \( y = \frac{1}{\alpha} (a^2 + b^2 + c^2 + d^2) \), \( z = \frac{1}{\alpha} (a^2 + b^2 - c^2 - d^2) \).

594. \( z_1 = \frac{1}{\alpha^2} \sum_{i=1}^{n} \left( a_i - a_1 \right) \), \( \sum_{i=1}^{n} \left( a_i - a_1 \right) \), where \( f(b) = \frac{1}{(b - a_k) b' f'(a_k)} \).

595. \( z = \frac{1}{\alpha^2} \sum_{i=1}^{n} \left( a_i - a_1 \right) \), \( \sum_{i=1}^{n} \left( a_i - a_1 \right) \), \( \left( a_i - a_1 \right) \) where \( f(b) = \frac{1}{(b - a_k) b' f'(a_k)} \).

596. \( z = \frac{1}{\alpha^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \left( b_i f(k) \right) \), where \( f(k) \) is the sum of all possible products of \( n - k \) numbers taken from the \( n - 1 \) numbers \( a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_n \).

where \( f(k) \) is the sum of all possible products of \( n - k \) numbers taken from the \( n - 1 \) numbers \( a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_n \).

597. \( x = \frac{1}{\alpha} p - h \), where \( p_i \) \( (i = 1, 2, \ldots, n - 1) \) is the sum of all possible products of \( i \) numbers taken from \( 1, 2, \ldots, n \) and \( P_0 = 1 \).

598. \( x = \frac{1}{\alpha} \sum_{i=1}^{n} \frac{b_i - a_i}{\alpha - a_i} \). Hint: Express the determinant of the system as a product of two determinants.

599. \( x = \frac{1}{\alpha} \sum_{i=1}^{n} \frac{b_i - a_i}{\alpha - a_i} \). Hint: Express the determinant of the system as a product of two determinants.

600. Solution. \( z = x - \frac{1}{2} x^3 + \frac{1}{4} x^5 - \frac{1}{4} x^7 + \ldots \). Therefore

\[ 1 + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \ldots \] and

\[ 1 - \frac{1}{2} x + \frac{1}{3} x^2 - \frac{1}{4} x^3 + \ldots \]

we obtain a system of equations for determining \( b_1, b_2, b_3, \ldots \). To prove that \( b_{n-1} = 0 \) when \( n > 1 \), note that \( b_1 = - \frac{1}{2} \) and that function

\[ f(x) = \frac{x^2}{e^x - 1} + 1 + \frac{1}{2} x = \frac{x^2}{2} + \frac{x^2}{e^x - 1} \]

is even.

603. Hint. Using the equations \( b_1 = - \frac{1}{2} \), \( b_{n-1} = -1 \) for \( n > 1 \), obtained in the preceding problem, set \( b_{n} = c_{n} \) and in the identity

\[ 1 - \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \ldots = \frac{1 - \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \ldots}{1 + \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \ldots} \]

equate separately the coefficients of even and odd powers of \( x \).

604. Hint. To establish the required equality, put \( x = 1, 2, \ldots, k - 1 \) in the identity

\[ (x + 1)^n = x^n + C_n^1 x^{n-1} + C_n^2 x^{n-2} + \ldots + C_n^k x^0 \]
and add the resulting equalities. In the equality thus established replace \( n \) by \( n-1, n-2, \ldots, 2, 1 \) and find \( a_{n-1} \) \((n)\) from the resulting system of \( n+1 \) linear equations in \( a_0, a_1, \ldots, a_{n-1} \).

605. \[
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}
\]

Hint. Obtain the identity

\[ x^n - x^{n-1} - \cdots - x \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) = \left( 1 + \frac{x}{2!} - \frac{x^3}{4!} + \cdots \right) + \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) \]

606. \[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1 \\
\end{pmatrix}
\]

607. \[
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & 1 & \cdots & 1 \\
\end{pmatrix}
\]

Hint. Use the identity

\[ a_0 + a_2 + a_4 + \cdots = \frac{a_0}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots} \]

608. 2, 609. 3, 610. 3, 611. 2, 612. For \( \lambda = 0 \) the rank of the matrix is 2, for \( \lambda \neq 0 \) it is equal to 3. 613. For \( \lambda = 3 \) the rank is equal to 2, for \( \lambda = -3 \) it is 3. 614. 2, 615. 2, 616. 3, 617. 2, 618. Hint. Using the linear expression of all columns of matrix \( A \) in terms of the columns that pass through the minor \( a_r \), show that if \( d = 0 \), the rows of the matrix \( A \) passing through \( d \) are linearly dependent.

619. If \( 0 \leq r \leq n \), then \( r = 0 \). If \( r = n-1 \), then \( r = 1 \).

620. If \( r = n \), then \( r = n \). Hint. Take advantage of problem 605 or 747.

621. Solution. We prove (1). For \( r = 0 \) all principal minors of the first and second orders are equal to zero. If \( A = (a_{ij}) \), then

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
\end{pmatrix}
= a_{11} a_{22} - a_{12} a_{21}
\]

for arbitrary \( i, j = 1, 2, \ldots, n \). From this we have \( a_{11} = 0 \) or \( a_{22} = 0 \). If \( i = 1 \), \( j = 1, 2, \ldots, n \); \( A = 0 \); the rank of \( A \) is zero, which is what we set out to prove. For \( r = n-1 \) we have \( M_{n-1} \neq 0 \), \( M_{n-2} = 1 \), \( |A| = 0 \) and the rank of \( A \) is equal to \( n-1 \).

Let \( 1 < r \leq n-2 \). The principal minor \( M_r \neq 0 \). By interchanging the rows and columns of matrix \( A \) this does not upset the symmetry of \( A \) after its rank, we can transfer the minor \( M_r \) to the upper left-hand corner of matrix \( A \). When proving (1), it suffices to show that all minors of order \( r+1 \) that border \( M_r \) are zero.

Let \( M_{ij} \) be a minor obtained from \( M_r \) by bordering it with the \( i \)th row and \( j \)th column, \( 1 \leq i, j \leq n \). By hypothesis, \( M_{ij} \neq 0 \) when \( i = j \). Let \( i \neq j \) and let \( B \) be the determinant obtained from \( M_{ij} \) by bordering it with the \( i \)th and \( j \)th rows and the \( i \)th and \( j \)th columns. By hypothesis, \( B = 0 \). Let \( C \) be the matrix of the determinant \( B \).

Suppose \( M_{ij} \neq 0 \). Then the rank of \( C \) is equal to \( r+1 \). Let \( r+1 \) rows and columns of matrix \( C \) with labels \( 1, 2, \ldots, r+1 \) are linearly independent. By the symmetry of \( C \), the columns with the same labels are also linearly independent. By problem 629, the minor \( M_r \) located at the intersection of three rows and columns is nonzero, but this runs counter to the hypothesis. Assertion (2) follows from (1) or directly from problem 629.

632. Hint. Use the preceding problem.


634. Hint. Take advantage of the preceding problem.

635. (1, 2, 3, 4), 636. \( x = (0, 1, 2, -3) \). 637. \( x = (0, 1, 2, 3, 4) \).


644. Linearly independent.

645. Hint. Assuming that \( \sum_{i=1}^{n} \lambda_i a_i = 0 \), where not all \( \lambda_i \) are zero, and taking, from among the \( \lambda_i \), the largest (in absolute value) coefficient \( \lambda_i \), show that the \( i \)th coordinate of the linear combination thus chosen is different from zero.

656. Hint. In the assumption that two vectors \( a_j, a_k \) \((i > j)\) can be expressed linearly in terms of the preceding ones, find an expression for the vector \( b \) from the expression of \( a_j \) and substitute that expression into the expression of \( a_k \).
In the set of vectors \( a_1, a_2, \ldots, a_n \), adjoin \( b \) on the left and delete the vectors that are linearly expressible in terms of preceding vectors; then adjoin \( b \) on the left and again delete the vectors that can be expressed linearly in terms of the preceding vectors, and so forth.

Take advantage of the preceding problem.

* Take advantage of the preceding problem.

When proving (3), make use of problems 663 and 664.

** Hint.** Make use of problems 663 and 664. Among the systems to be ordered, adjoin to the right of the system all vectors of the system and delete all vectors that can be linearly expressed in terms of the preceding ones.

663. It is impossible to choose such numbers.

664. **Hint.** Make use of problem 663 and also problem 665 (from (c)).

665. \( \lambda = 15 \). 665. \( \lambda \) is any number. 667. \( \lambda \) is any number.

668. \( \lambda \) is not equal to 12. 669. There is no such value of \( \lambda \).

670. The vectors \( a_1, a_2, a_3 \) in problem 668 are coplanar (lie in one plane) but are not collinear (they do not lie on one line). When \( \lambda = 15 \), vector \( b \) lies in the same plane and is expressed in terms of \( a_1, a_2, a_3 \), but when \( \lambda \neq 15 \), it does not lie in that plane and cannot be expressed in terms of these vectors. In problem 666 the vectors \( a_1, a_2, a_3 \) are not coplanar and any vector of three-dimensional space can be expressed linearly in terms of them. In problem 667 the vectors \( a_1, a_2, a_3 \) are not collinear and lie in the plane \( 4x - 3z = 0 \).

671. For any value of \( \lambda \), the vector \( b \) lies in the same plane and can be expressed linearly in terms of \( a_1, a_2, a_3 \).

In problem 668, the vectors \( a_1, a_2, a_3 \) are not collinear and the vector \( b \) is not coplanar with \( a_1, a_2, a_3 \). When \( \lambda = 12 \), the vectors \( a_1, a_2, a_3 \) are not coplanar and can be expressed in terms of them.

672. There are four such systems: (1) \( a_1, a_3, a \); (2) \( a_1, a_4, a \); (3) \( a_2, a_3, a \); (4) \( a_2, a_4, a \).

673. Any two vectors form a basis.

674. (1) \( a_1, a_2, a_3 \); (2) \( a_1, a_4, a_3 \); (3) \( a_2, a_3, a_4 \); (4) \( a_2, a_4, a_3 \).

675. Any three vectors, except \( a_1, a_2, a_3 \) and \( a_2, a_3, a_4 \), form a basis.

676. The only basis occurs if and only if either the entire system or the basis of vectors of the system that do not appear in the basis.

677. In the set of vectors \( a_1, a_2, a_3, a_4, a_5 \), adjoin \( b \) on the left and delete the vectors that are linearly expressible in terms of preceding vectors; then adjoin \( b \) on the left and again delete the vectors that can be expressed linearly in terms of the preceding vectors, and so forth.

678. **Hint.** Make use of problems 663 and 667.

679. **Hint.** Make use of the given system to be ordered, adjoin to the right of the system all vectors of the system and delete all vectors that can be linearly expressed in terms of the preceding ones.

680. It is impossible to choose such numbers.

681. **Hint.** Make use of problem 680 and also problem 682 (from (4)).

682. **Hint.** When proving (4), put into

\[ \sum_{i=1}^{n} \alpha_i x_i = 0, \]

the expression \( \sum_{i=1}^{n} \alpha_i x_i = 0 \) and show that \( \alpha_1 = -\sum_{i=1}^{n} \alpha_i x_i \).

Using this fact, show that \( \alpha_1 = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \), where the coordinate equal to unity occupies the \((r+1)\)th position and is \(-x_1 x_2 x_3 \cdots x_n \), then \( \sum_{i=1}^{n} \alpha_i x_i = 0 \).

683. **Hint.** When proving (2) to (4), make use of problem 684.

684. \( f_1 = f_2 = 0 \), \( f_1 = f_2 = 0 \). When proving (2) to (4), make use of problem 684.

685. The forms are linearly independent. No basic system of linear relations exists.

686. \( f_1 = f_2 = f_3 = 0 \) for example, the general solution is \( x_1 = x_2 = x_3 = 1 \).

**Hint.** Make use of problem 684 or 687.

687. The systems are linearly independent. No basic system of linear relations exists.

688. \( f_1 = f_2 = f_3 = 0 \).

For example, the general solution is \( x_1 = x_2 = x_3 = 1 \).

**Hint.** Make use of problem 684 or 687.

689. For example, the general solution is \( x_1 = x_2 = x_3 = 1 \).

690. For example, the general solution is \( x_1 = x_2 = x_3 = 1 \).

691. General solution: \( x_1 = x_2 = x_3 = 1 \).

692. System is inconsistent.

693. The system has a unique solution: \( x_1 = 3, x_2 = 4, x_3 = 1 \).

694. General solution: \( x_1 = x_2 = 1 \), \( x_3 = 1 \).

695. General solution: \( x_1 = x_2 = x_3 = 1 \).

696. General solution: \( x_1 = x_2 = x_3 = 1 \).

697. General solution: \( x_1 = x_2 = x_3 = 1 \).

698. General solution: \( x_1 = x_2 = x_3 = 1 \).
A particular solution: 
\[ z_1 = -2, \quad z_2 = 2, \quad z_3 = 3, \quad z_4 = -1. \]

The system is inconsistent.

A general solution: 
\[ z_1 = -1 - 8x_1 + 4x_2, \quad z_4 = 0, \quad z_5 = 1 + \frac{\bar{x}_1}{2} - x_2. \]

A particular solution: 
\[ z_1 = 1, \quad z_2 = 2, \quad z_3 = -1, \quad z_4 = 0, \quad z_5 = -1. \]

The system is inconsistent.

A particular solution: 
\[ z_3 = 13, \quad z_4 = 19 - 3z_1 - 2x_2, \quad z_5 = -34. \]

A general solution: 
\[ z_1 = 1, \quad z_2 = 8, \quad z_3 = 13, \quad z_4 = 0, \quad z_5 = -34. \]

The system is inconsistent.

A particular solution: 
\[ z_3 = 1, \quad z_4 = 2t, \quad m = -34. \]
is of the form \( x = \frac{b - d}{b - a} = 1, \ y = \frac{b - c}{b - a} \), where \( z \) is a free unknown that plays the part of the above-mentioned parameter which determines the solution.

If \( a = b = c = d \), then the solution depends on two parameters and the general solution is, say, of the form \( x = 1 - y - z \), where \( y, z \) are free unknowns. If there are two distinct numbers from among \( a, b, c \) and if \( d \) is not equal to any one of them or if \( a = b = c \neq d \), then the system is inconsistent.

722. When \( D = abc - a - b - c + 2 \neq 0 \) the system has a unique solution:

\[
x = \frac{(b-1)(c-1)}{D}, \quad y = \frac{(a-1)(c-1)}{D}, \quad z = \frac{(c-1)(b-1)}{D}.
\]

In this case, any two of the unknowns can have zero values at the same time, while the third unknown and the corresponding parameter are equal to unity. For example, \( x = y = 0, z = 1 \). If \( D = 0 \), and if one and only one of the numbers \( a, b, c \) is different from unity, then the solution depends on a single parameter: for example, when \( a \neq b = c = 1 \) the general solution is of the form \( x = 0, y = 1 - z \). In this case, one or two of the unknowns must be zero.

If \( a = b = c = 1 \) the general solution is of the form \( x = 1 - y - z \) and one or two of the unknowns may be zero. If \( D = 0 \) and none of the numbers \( a, b, c \) is equal to unity, then the system is inconsistent. The case of \( D = 0 \) with one and only one of the numbers \( a, b, c \) equal to unity is impossible.

723. If \( D = abc - a - b - c + 2 \neq 0 \), the system has a unique solution:

\[
x = \frac{abc - 2bc + b + c - b}{D}, \quad y = \frac{abc - 2ac + a + c - b}{D}, \quad z = \frac{abc - 2ab + a + b - c}{D}.
\]

If \( D = 0 \) and only one of the numbers \( a, b, c \) is different from unity, then the solution depends on a single parameter, for example, when \( a \neq b = c = 1 \) the general solution is of the form \( x = 1, y = 1 - z \), where \( z \) is a free unknown. If \( a = b = c = 1 \), then the solution depends on two parameters and the general solution is of the form \( x = 1 - y - z \), where \( y, z \) are free unknowns. If \( D = 0 \), and all numbers \( a, b, c \) are different from unity, the system is inconsistent. The case of \( D = 0 \) with only one of the numbers \( a, b, c \) equal to unity is impossible. Hint. In order to prove the inconsistency of the system when \( D = 0 \) under the condition that all numbers \( a, b, c \) are different from unity, show that the following identities hold true:

\[
D = D_x = 2 (b - 1)(c - 1), \quad D = D_y = 2 (a - 1)(c - 1),
\]

\[
D = D_z = 2 (a - 1)(b - 1),
\]

where \( D_x, D_y, D_z \) are the appropriate numerators in the above expressions for \( x, y, z \).

724. For example, the general solution is \( x_1 = 3x_2 + x_3, \ x_2 = 1, \ x_3 = -6x_2 + 5x_4 \). The fundamental set of solutions is

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

725. General solution: \( x_3 = -\frac{5}{2} x_1 + 5x_2, \ x_4 = 7 x_2 - 7x_3 \). The fundamental solution set is

\[
\begin{array}{cccc}
x_1 & x_2 & x_3 & x_4 \\
1 & 0 & -\frac{5}{2} & +\frac{7}{2} \\
0 & 1 & +5 & -7 \\
\end{array}
\]

726. General solution: \( x_4 = -\frac{3x_3 + 6x_2 + 3x_1}{4} \), \( x_3 = \frac{3x_3 + 2x_2 + 4x_1}{4} \). The fundamental solution set is

\[
\begin{array}{cccc}
x_1 & x_2 & x_3 & x_4 \\
1 & 0 & 0 & -\frac{9}{4} \\
0 & 1 & 0 & -\frac{3}{2} \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

727. The system only has a zero solution. There is no fundamental solution set.

728. General solution: \( x_4 = -\frac{9x_3 + 3x_2 - 10x_1}{15} \), \( x_3 = -\frac{3x_1 + 2x_2 + 4x_4}{11} \). The fundamental solution set is

\[
\begin{array}{cccc}
x_1 & x_2 & x_3 & x_4 \\
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{array}
\]
729. The system has only a zero solution.

730. General solution: \( x_1 = x_4 - x_5, x_2 = x_8 - x_9, x_3 = x_4. \) The fundamental solution set is

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

731. General solution: \( x_1 = 0, x_5 = \frac{x_3 - 2x_5}{3}, x_4 = 0, \) where \( x_3, x_5 \) are free unknowns. The fundamental solution set is

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

732. General solution: \( x_1 = -3x_3 - 5x_5, x_2 = 2x_3 + 3x_5, x_4 = 0, \) where \( x_3, x_5 \) are free unknowns. The fundamental solution set is

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-5</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

735. \( x_1 = 13c, x_2 = 2c, x_3 = 7c, x_4 = c_1 - c_2, x_5 = c_3 - c_4, x_6 = c_5 - c_6. \)

736. \( x_1 = 4c_1, x_2 = 8c_2, x_3 = -3c_3 + 3c_4, x_5 = c_1 - c_2, x_6 = c_3 - c_4. \)

737. \( x_1 = 2c_1, x_2 = c_2, x_3 = 3c_3, x_4 = -3c_1 - c_2 + 4c_3, x_5 = -4c_1. \)

738. \( x_1 = 2c_1, x_2 = 3c_2, x_3 = 2c_3, x_4 = -3c_1 + 3c_2 - 8c_3, x_5 = -6c_1 + 8c_2 - 10c_3. \)

739. \( x_1 = c_1 - c_2, x_2 = 2c_1 + 6c_2, x_3 = c_1 + 3c_2, x_4 = -4c_1, x_5 = -6c_2. \)

740. \( x_1 = 11c_1, x_2 = 33c_2, x_3 = -24c_1 - 57c_2, x_4 = 5c_1 + 5c_2, x_5 = -3c_1 - 3c_2. \)

741. The rows of matrix \( A \) do not form a fundamental solution set, the rows of matrix \( B \) do.

742. The fourth row together with any two of the first three rows forms a fundamental set, while the other systems of rows do not form a fundamental set.

743. Hint. In the first part of the problem apply the result of problem 734. In the second part of the problem show that if the values of the free unknowns yield, in a certain solution set, linearly dependent rows, then the whole solution set is linearly dependent.

748. Hint. Adjoin to the top of the matrix of the system any one of the rows and then expand the determinant of the resulting matrix in terms of the first row. Make use of problem 746.

749. A particular solution: \( x_1 = -2, x_2 = -6, x_3 = 0, x_4 = 1. \) The general solution is \( x_1 = 3c, x_2 = -2c, x_3 = 0. \)

750. A particular solution: \( x_1 = -2, x_2 = -6, x_3 = 0, x_4 = 1. \) The general solution is \( x_1 = 3c, x_2 = -2c, x_3 = 0. \)

751. A particular solution: \( x_1 = 0, x_2 = 11, x_3 = -9, x_4 = 4. \) The general solution is \( x_1 = 4c, x_2 = 0, x_3 = 4, x_4 = 0. \) The general solution is \( x_1 = 3c, x_2 = 0, x_3 = 4c, x_4 = 0. \)

754. (a) \( \sum_{j=1}^{n} a_{ij}x_j = 2b_i \) \( (i = 1, 2, \ldots, s). \)

(b) \( \sum_{j=1}^{n} a_{ij}x_j = \lambda b_i \) \( (i = 1, 2, \ldots, s). \)

755. In both cases the necessary and sufficient condition is the homogeneity of the given system.

756. Provided that the sum of the coefficients of the given linear combination is equal to unity.

757. The first unknown in any solution assumes the value zero. If the coefficients of all unknowns, except the first and, for example, the second, are equal to zero, then the second unknown takes on a definite value that can be found from an equation involving a nonzero coefficient of the second unknown if all terms involving other unknowns are dropped; in that case, all unknowns from the third onwards may assume arbitrary values. But if at least three unknowns (say \( x_1, x_2, \) and \( x_3 \)) occur with nonzero coefficients, then all unknowns, except the first, may assume arbitrary values, and their values in each solution are connected by a single relation obtained from any equation of the system that contains a nonzero coefficient of the second unknown, if the term involving the first unknown is dropped.
Under the conditions of the problem, it is impossible that all coefficients of the first unknown or of all unknowns, beginning with the second, can be equal to zero.

758. A necessary and sufficient condition for this is that the rank of the matrix of coefficients of the unknowns be reduced by unity when the kth column is deleted, in other words that the kth column not be a linear combination of the other columns of the matrix.

759. The rank of the augmented matrix (made up of the coefficients of the unknowns and the constant terms) must be reduced by unity when the kth column is deleted.

760. The rank of the augmented matrix (made up of the coefficients of the unknowns and the constant terms) must be reduced by unity when the kth column is deleted.

761. One condition stating that the determinant \( D \) of order \( r \) is not equal to zero, and \( \left( x - i \right) \left( x - r + 1 \right) \) conditions stating that the determinants of order \( r + 1 \) of the bordering rows and columns containing each contain an element that does not appear in the other conditions; it stands at the intersection of the bordering row and column and has a factor \( D = 0 \).

762. Either at least two of the numbers \( a, b, c, d, e \) are equal to \(-1\) or none is equal to \(-1\), but then

\[
\frac{a}{x+1} + \frac{b}{y+1} + \frac{c}{z+1} + \frac{d}{w+1} + \frac{e}{v+1} = 1.
\]

763. \( \lambda = ax + by + cz = 0 \). Hint. Add all equations after multiplying them, respectively, by \( \lambda x, \lambda y, \lambda z, \lambda t \). After obtaining the condition \( \lambda = 0 \), the determinant of the system may be computed as a skew-symmetric determinant following problem 547.

764. \[
\begin{vmatrix}
1 & x_1 & y_1 & 1 \\
x_2 & x_2 & y_2 & 1 \\
x_3 & x_3 & y_3 & 1 \\
\end{vmatrix} = 0.
\]

765. \[
\begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1 \\
\end{vmatrix} = 0.
\]

766. \[
\begin{vmatrix}
a_1 & b_1 & c_1 & 1 \\
a_2 & b_2 & c_2 & 1 \\
a_3 & b_3 & c_3 & 1 \\
\end{vmatrix} = 0;
\]

this condition is sufficient if, in the case of three parallel lines, their common point is taken to be the ideal point (point at infinity) of the given direction. If ideal points are excluded, then the necessary and sufficient condition is the equality of the ranks of two matrices:

\[
\begin{pmatrix}
a_1 & b_1 & c_1 & 1 \\
a_2 & b_2 & c_2 & 1 \\
a_3 & b_3 & c_3 & 1 \\
\end{pmatrix}
\]

767. The rank of the matrix

\[
\begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1 \\
\end{vmatrix}
\]

must be less than three.

768. Assuming ideal points, the rank of the matrix

\[
\begin{pmatrix}
a_1 & b_1 & c_1 & 1 \\
a_2 & b_2 & c_2 & 1 \\
a_3 & b_3 & c_3 & 1 \\
a_4 & b_4 & c_4 & 1 \\
\end{pmatrix}
\]

must be less than three. If we do not assume ideal points, the rank of the reduced matrix must coincide with the rank of the matrix

\[
\begin{pmatrix}
a_1 & b_1 & 1 \\
a_2 & b_2 & 1 \\
a_3 & b_3 & 1 \\
a_4 & b_4 & 1 \\
\end{pmatrix}
\]

771. \( x^2 + y^2 = 4x - 1 = 0 \). The centre is at the point \( (2, 0) \).

The radius is equal to \( \sqrt{5} \).

772. Hint. Use the answer to problem 570.

773. \[
\begin{vmatrix}
x^3 & xy & y^2 & z & y \\
x_1 & x_1 & y_1 & y_1 & 1 \\
x_2 & x_2 & y_2 & y_2 & 1 \\
x_3 & x_3 & y_3 & y_3 & 1 \\
\end{vmatrix} = 0.
\]

774. The hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \).

775. \( 2x^2 + 7y^2 + y - 3 = 0 \). This is an ellipse with centre at the point \( \left( \frac{1}{14}, 0 \right) \) and with semi-axes of length \( \frac{1}{14} \sqrt{28} \) and \( \frac{1}{14} \sqrt{5} \); the major axis is parallel to the axis of abscissas and the minor axis lies on the axis of ordinates.

776. \[
\begin{vmatrix}
x_1 & y_1 & z_1 & 1 \\
x_2 & x_2 & y_2 & 1 \\
x_3 & x_3 & y_3 & 1 \\
\end{vmatrix} = 0.
\]

777. \[
\begin{vmatrix}
x & y & z & 1 \\
1 & 1 & 1 & 1 \\
2 & 3 & -4 & 1 \\
3 & -1 & -1 & 1 \\
\end{vmatrix} = 0.
\]

778. If one assumes ideal points (points at infinity), then

\[
\begin{vmatrix}
a_1 & b_1 & c_1 & d_1 \\
a_2 & b_2 & c_2 & d_2 \\
a_3 & b_3 & c_3 & d_3 \\
a_4 & b_4 & c_4 & d_4 \\
\end{vmatrix} = 0.
\]
If one does not assume ideal points, the rank of the matrix
\[
\begin{pmatrix}
    a_1 & b_1 & c_1 & d_1 \\
    a_2 & b_2 & c_2 & d_2 \\
    a_3 & b_3 & c_3 & d_3 \\
    a_4 & b_4 & c_4 & d_4 \\
\end{pmatrix}
\]
must not change when the last column is deleted.

779. The rank of the matrix
\[
\begin{pmatrix}
    a_1 & b_1 & c_1 & d_1 \\
    a_2 & b_2 & c_2 & d_2 \\
    a_3 & b_3 & c_3 & d_3 \\
    a_4 & b_4 & c_4 & d_4 \\
\end{pmatrix}
\]
is equal to two and does not change when the last column is deleted.

780. \[
x^2 + y^2 + z^2 = 0.
\]
The centre lies at the point \((1, 0, 0)\). The radius is equal to 3/2.

782. A system of three linear equations in two unknowns; in this system, the augmented matrix made up of the coefficients of the unknowns for any pair of equations all have a rank of 2.

783. A system of three linear equations in two unknowns; in this system, the ranks of the matrices of the coefficients of the unknowns in any pair of equations are equal to two, and the rank of the augmented matrix is equal to three.

784. A system of three equations in three unknowns; in this system, the ranks of all matrices made up of the coefficients of the unknowns of any two equations and also of all three equations are equal to two, and the rank of the augmented matrix is equal to three.

785. A system of four linear equations in three unknowns; in this system, the ranks of the matrices made up of the coefficients of the unknowns of any three equations are equal to three, and the rank of the augmented matrix is equal to 4.

786. Four planes pass through one point, but no three of them through one straight line.

787. If one does not consider ideal straight lines and planes (lines and planes at infinity), then equations of the form \(ax + by = c\) are geometrically meaningless; but when \(c = 0\) they are satisfied by the coordinates of any point in the plane or in space. By eliminating equations of that type designating the rank of the matrix made up of the coefficients of the unknowns by \(r\) and the rank of the augmented matrix by \(r_1\), we have the following:

For systems in two unknowns:
1. \(r = 2, r_1 = 3\). The system has no solutions. The straight lines do not pass through one point, and at least two straight lines are distinct and intersect.
2. \(r = r_1 = 2\). The system has a unique solution. The straight lines pass through one point, and at least two are distinct.
3. \(r = 1, r_1 = 2\). The system has no solutions. The straight lines are parallel or coincident, and at least two straight lines are distinct.
4. \(r = r_1 = 1\). The solution depends on one parameter. All straight lines are coincident.

For systems in three unknowns:
1. \(r = 3, r_1 = 4\). The system has no solutions. The planes do not pass through one point, and at least three of them are distinct and pass through one point.
2. \(r = r_1 = 3\). The system has a unique solution. The planes pass through one point, and at least three of them do not pass through one straight line.
3. \(r = 2, r_1 = 3\). The system has no solutions. The planes do not pass through one point, and at least three planes are distinct and any three distinct planes either have no point in common or pass through one straight line.
4. \(r = r_1 = 2\). The solution depends on one parameter. All planes pass through one straight line, and at least two of them are distinct.
5. \(r = 1, r_1 = 3\). The system has no solutions. The planes are parallel or coincide, and at least two of them are distinct.
6. \(r = r_1 = 1\). The solution depends on two parameters. All planes are coincident.

Chapter III. Matrices and Quadratic Forms

788. \[
\begin{pmatrix}
    5 & 2 \\
    7 & 0
\end{pmatrix}
\]
789. \[
\begin{pmatrix}
    a & x + b y + c z \infty \\
    c & d + e y + f z + g z \infty
\end{pmatrix}
\]
790. \[
\begin{pmatrix}
    1 & 5 & -5 \\
    3 & 10 & 0
\end{pmatrix}
\]
791. \[
\begin{pmatrix}
    11 & -2 & 29 \\
    9 & -27 & 32
\end{pmatrix}
\]
792. \[
\begin{pmatrix}
    10 & 17 & 19 & 23 \\
    17 & 23 & 27 & 35
\end{pmatrix}
\]
793. \[
\begin{pmatrix}
    8 & 6 & -5 & 2 \\
    13 & -17 & 26
\end{pmatrix}
\]
794. \[
\begin{pmatrix}
    0 & 0 \\
    0 & 0
\end{pmatrix}
\]
795. \[
\begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]
796. \[
\begin{pmatrix}
    0 & 0 \\
    0 & 0
\end{pmatrix}
\]
802. \( \cos n\alpha - \sin n\alpha \), \( \sin n\alpha \cos n\alpha \).

803. \[
\begin{pmatrix}
\lambda^0 & 0 \\
\vdots & \ddots \\
0 & \lambda^n
\end{pmatrix}
\]

804. \[
\begin{pmatrix}
n \\
0
\end{pmatrix}
\]

805. \[
\begin{pmatrix}
n^2 & n & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

806. \[
\begin{pmatrix}
1 & 3 & 5 & \ldots & \frac{n(n+1)}{2} \\
0 & 1 & 3 & \ldots & \frac{(n-1)n}{2} \\
0 & 0 & 1 & \ldots & \frac{(n-2)(n-1)}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

where \( n \) is the order of the given matrix.

807. \[
\begin{pmatrix}
1 & C_{n-1}^1 & C_{n-1}^2 & \ldots & C_{n-1}^{n-1} \\
0 & 1 & C_{n-1}^2 & \ldots & C_{n-1}^{n-2} \\
0 & 0 & 1 & \ldots & C_{n-1}^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

808. \[
\begin{pmatrix}
2197 & -1268 \\
7385 & -922
\end{pmatrix}
\]

809. \[
\begin{pmatrix}
190 & 189 & -189 \\
126 & 127 & -126 \\
252 & 252 & -251
\end{pmatrix}
\]

811. (a) The \( i \)th and \( j \)th rows of the product are interchanged; (b) the \( st \)th row of the product is added the \( i \)th row multiplied by \( c \); (c) the \( i \)th and \( j \)th columns of the product are interchanged; (d) to the \( st \)th column of the product is added the \( j \)th column multiplied by \( c \).
836. \[
\begin{pmatrix}
-2 & 1 \\
3/2 & -1/2
\end{pmatrix}
\]
837. \[
\begin{pmatrix}
7 & -4 \\
-5 & 3
\end{pmatrix}
\]
838. \[
\frac{1}{ad-bc} \begin{pmatrix}
d & -b \\
-c & a
\end{pmatrix}
\]
839. \[
\begin{pmatrix}
cos \alpha & -sin \alpha \\
-sin \alpha & cos \alpha
\end{pmatrix}
\]
840. \[
\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]
841. \[
\begin{pmatrix}
-8 & 29 & -41 \\
-5 & 13 & -7
\end{pmatrix}
\]
842. \[
\begin{pmatrix}
-7/3 & 2 & -1/3 \\
5/3 & -1 & -1/3
\end{pmatrix}
\]
843. \[
\frac{1}{9} \begin{pmatrix}
1 & 2 \\
3 & 1
\end{pmatrix}
\]
844. \[
\begin{pmatrix}
1 & 1 \\
1 & -1 \\
1 & 1 \\
1 & -1
\end{pmatrix}
\]
845. \[
\begin{pmatrix}
22 & -6 & -26 & 17 \\
-17 & 5 & 20 & -13 \\
-1 & 0 & 2 & -1 \\
4 & -1 & -5 & 3
\end{pmatrix}
\]
846. \[
\begin{pmatrix}
1 & 1 \\
1 & -1 \\
1 & 1 \\
1 & -1
\end{pmatrix}
\]
847. \[
\begin{pmatrix}
1 & 1 & 1 & \ldots & (\frac{1}{2})^{n-1} \\
0 & 1 & 1 & \ldots & (\frac{1}{2})^{n-2} \\
0 & 0 & 1 & \ldots & (\frac{1}{2})^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}
\]
848. \[
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]
849. \[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}
\]
850. \[
\begin{pmatrix}
1 & -2 & 1 & 0 & \ldots & 0 \\
0 & -2 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & -2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & -2 \\
\end{pmatrix}
\]
851. \[
\frac{1}{n+1} \begin{pmatrix}
1 & 2 & 3 & \ldots & 1 \\
2 & 1 & 2 & \ldots & 1 \\
3 & 2 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n-3 & 2 & 1 & \ldots & 1 \\
n-2 & 2 & 1 & \ldots & 1 \\
n-1 & 2 & 1 & \ldots & 1 \\
1 & 2 & 3 & \ldots & n
\end{pmatrix}
\]
852. \[
\frac{1}{n-1} \begin{pmatrix}
1 & 2 & \ldots & 1 \\
1 & 2 & \ldots & 1 \\
1 & 2 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2 & \ldots & 1
\end{pmatrix}
\]
853. \[
\frac{1}{a(n+a)} \begin{pmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{pmatrix}
\]
854. \[
\frac{1}{s} \begin{pmatrix}
1 & -a & 1 & \ldots & 1 \\
1 & 1 & -a & \ldots & 1 \\
1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & 1
\end{pmatrix}
\]
855. \[
\begin{pmatrix}
\frac{1-a}{a^2} & \frac{1-a}{a^3} & \frac{1-a}{a^4} & \ldots & \frac{1}{a^{n-1}} \\
\frac{1-a}{a^2} & \frac{1-a}{a^3} & \frac{1-a}{a^4} & \ldots & \frac{1}{a^{n-1}} \\
\frac{1-a}{a^2} & \frac{1-a}{a^3} & \frac{1-a}{a^4} & \ldots & \frac{1}{a^{n-1}} \\
\frac{1-a}{a^2} & \frac{1-a}{a^3} & \frac{1-a}{a^4} & \ldots & \frac{1}{a^{n-1}} \\
\frac{1-a}{a^2} & \frac{1-a}{a^3} & \frac{1-a}{a^4} & \ldots & \frac{1}{a^{n-1}}
\end{pmatrix}
\]
where \( s = 1 + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}. \)
In the system of equations, for elements of the kth column of the inverse matrix, subtract from each equation from the first to the (n — 1)th—the next equation and then add the resulting n — 1 equations. Express all unknowns in terms of the kth.

\[ s = m(a + h) \]

where \( s = na + h \), is the sum of the elements of some row (or column) of the given matrix.

Write the equations with unknowns \( x_1, x_2, \ldots, x_n \) in order to determine the elements of the kth column of the inverse matrix. Multiply each equation by a power of \( e \) such that the coefficient of a definite unknown \( x_j \) becomes unity. Add the resulting equations.

\[ \frac{1}{a^2} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & e^{n-1} & e^{n-2} & \ldots & e^{n-1} \\ 1 & e^{n-2} & e^{n-3} & \ldots & e^{(n-1) - 1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{n-1} & e^{n-2} & \ldots & e^{n-2} \\ 1 & e^{n-2} & e^{n-3} & \ldots & e^{n-3} \end{pmatrix} \]

\[ \frac{1}{a^2} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & e^{n-1} & e^{n-2} & \ldots & e^{n-1} \\ 1 & e^{n-2} & e^{n-3} & \ldots & e^{(n-1) - 1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{n-1} & e^{n-2} & \ldots & e^{n-2} \\ 1 & e^{n-2} & e^{n-3} & \ldots & e^{n-3} \end{pmatrix} \]

\[ \frac{1}{a^2} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & e^{n-1} & e^{n-2} & \ldots & e^{n-1} \\ 1 & e^{n-2} & e^{n-3} & \ldots & e^{(n-1) - 1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{n-1} & e^{n-2} & \ldots & e^{n-2} \\ 1 & e^{n-2} & e^{n-3} & \ldots & e^{n-3} \end{pmatrix} \]

The general aspect of the solution is \( \left( \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_n \end{array} \right) \), where \( c_1 \) and \( c_n \) are arbitrary numbers.
901. Hint. Use the identity of problem 504.
904. Hint. Use the identity of problem 507.
905. The diagonal elements are equal to \pm 1.
906. The diagonal elements are equal to unity in absolute value.
908. or apply the hint given in that problem.
915. Hint. Take advantage of the preceding problem.
920. Hint. Take advantage of problem 913.
921. Hint. Use the Laplace theorem, the Cauchy-Bunyakovsky inequality and the Binet-Cauchy formula (see problems 506 and 499).
922. Hint. Let \( n \) be the number of rows of \( A, B, C, k \) the number of columns of \( B, C \), and \( l \) the number of columns of \( C \). Check that the inequality holds for \( k + l > n \) and that the rank of \( A < k + l < n \) and under the supplementary condition \( B' - C = 0 \). Finally, when the rank of \( A = k + l < n \), complete \( A \) to a square matrix \( (A, D) = P = (B, Q) \), where \( Q = (C, D) \), with the aid of \( n - k - l \) linearly independent columns so that \( A' D = 0 \) (this can be done by constructing a fundamental set of solutions of a homogeneous system of equations with matrix \( A' \)), apply the preceding case to the matrices \( P = (A, D) \) and \( Q = (C, D) \), and take into account that \( P = (B, Q) \) and \( D' D > 0 \).
923. Hint. Apply the inequality of the preceding problem several times.
924. Hint. Repeatedly apply the inequalities of problem 922.
925. Hint. Reason as indicated in the hint to problem 922.
926. Hint. Make repeated use of the inequality of the preceding problem and take advantage of the answer to problem 532.
927. The \( i \)th and \( j \)th rows or the \( i \)th and \( j \)th columns are interchanged when multiplied by a matrix whose elements \( p_{kh} = 1 \) for \( k \neq i \) and \( p_{ji} = p_{kj} = 1 \), and otherwise zero.
928. The \( i \)th row (or column) is multiplied by a number \( c \neq 0 \) when we have multiplication by a matrix that differs from the unit matrix only in the \( i \)th element of the principal diagonal \( a_{ii} \) equal to \( c \). Addition of the \( i \)th row multiplied by \( c \) to the \( j \)th row results when premultiplying by a matrix that differs from the unit matrix only in that the element \( p_{ij} = c \).
929. Hint. To determine the type of desired matrices, perform the elementary transformation with respect to the unit matrix in order equal to the number of rows of matrix \( A \) in the transformation of rows and to the number of columns of \( A \) in the transformation of columns. Verify that the resulting matrices satisfy the requirements of the problem.
928. Hint. Use problem 927 and show that transformation of type \( (b) \) is performed by several transformations of type \( (a) \). Take advantage of problems 917 and 927.
929. Hint. Use problem 923.
930. (24) \(-4 -8\)
931. \((-23/2 -1 7/2)\)
932. \((10 1 -2 -3)\)
933. \((-5 0 1 1)\)
934. \((-1/6 1/2 -7/6 10/3)\)
935. \((-7/6 -1/2 5/6 -5/3)\)
936. \((3/2 1/2 -4/2 1)\)
937. \((1/2 1/2 -1/2 1)\)
938. Hint. To prove necessity, take for \( B \) an nonzero column of matrix \( A \).
939. Hint. To prove necessity, take for \( B \) the matrix of any \( r \) linearly independent columns of matrix \( A \); make up the \( i \)th column of matrix \( C \) from coefficients in the expression of the \( i \)th column of \( A \) in terms of the columns of \( B \). Use problem 914.
941. Hint. Use integral elementary transformations to bring the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least the given matrix \( A \) to a normal matrix. To do this, choose the least
of matrix $C$ to the right of the main diagonal are equal to zero, that is, $C = E$.


949. Solution. Reduce the matrix $A$ to the upper triangular matrix $C$ via the following elementary operations: $a_{i1} = a_{i1} \neq 0$. Subtracting the first row multiplied by suitable numbers from the other rows, reduce $A$ to the form

$$A^1 = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ 0 & a_{32} & \ldots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \ldots & a_{nn} \end{pmatrix};$$

since the minors involving the first row do not change in this process, we have $c_{i1} = a_{i1} b_{i1} \neq 0$, whence $a_{12} \neq 0$. Subtracting the second row of matrix $A^1$ with suitable factors from the rows below, make the second elements of these rows vanish, and so on. After $r$ steps, all elements of the first $r$ columns lying below the diagonal will vanish.

Since the rank of the resulting matrix $A^{(r)} = C$ is equal to the rank of $A$, that is, equal to $r$, it follows that all elements of the last $n - r$ rows of matrix $A$ are zero and, hence, $C$ is an upper triangular matrix. By virtue of problem 927, $C = P A$, where $P$ is a product of a series of lower triangular matrices, that is, again, a lower triangular matrix. This proves the existence of an expansion of the type (2).

Suppose we have any representation of the type (2). By the formula for minors of a product of matrices (problem 913) we have

$$A = \begin{pmatrix} 1, & 2, & \ldots, & k-1, & i \\ 1, & 2, & \ldots, & k-1, & j \end{pmatrix} B = \begin{pmatrix} 1, & 2, & \ldots, & k-1, & i \\ 1, & 2, & \ldots, & k-1, & j \end{pmatrix} C = \begin{pmatrix} 1, & 2, & \ldots, & k-1, & k \\ 1, & 2, & \ldots, & k-1, & k \end{pmatrix}.$$

But the first $k$ columns of matrix $C$ contain only one $k$th-order minor different from zero, and for that reason

$$A = \begin{pmatrix} 1, & 2, & \ldots, & k-1, & i \\ 1, & 2, & \ldots, & k-1, & k \end{pmatrix} B = \begin{pmatrix} 1, & 2, & \ldots, & k-1, & i \\ 1, & 2, & \ldots, & k-1, & k \end{pmatrix} C = \begin{pmatrix} 1, & 2, & \ldots, & k-1, & k \\ 1, & 2, & \ldots, & k-1, & k \end{pmatrix}.$$

Setting $i = k$ here, we get

$$d_k = b_{1k} b_{2k} \ldots b_{kk} c_{2k} \ldots c_{kk} (k = 1, 2, \ldots, r). \tag{a}$$

Dividing (b) by a similar equality with $k$ replaced by $k - 1$, we get (3). Dividing (a) by (b), we get the first of the formulas (4). The second formula is obtained in similar fashion.

Let $S$ be any nonsingular diagonal matrix with elements $d_1, d_2, \ldots, d_n$ on the main diagonal. Then $A = BC = (BD)^{-1}$. The matrix $BD$ is obtained from $B$ by multiplying the columns by $d_1, d_2, d_3, \ldots, d_n$. The matrix $D^{-1} C$ is obtained from $C$ by multi-

951. Hint. Use the solution of problem 949 and show that under the conditions $b_{kk} = c_{kk} = \sqrt{d_{kk}}$, the conditions (3) are satisfied and the conditions (4) yield $b_{ij} = c_{ij}$ ($t = k = 1, k + 2, \ldots, n, k = 1, 2, \ldots, r$).

952. $AB = C = (C_{ij})$, where $C_{11} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$, $C_{12} = \begin{pmatrix} 2 & 4 \\ 9 & 6 \end{pmatrix}$, $C_{31} = \begin{pmatrix} 8 & 4 \\ 9 & 1 \end{pmatrix}$.

954. Hint. Consider the product of the $i$th block row by the $j$th block column.

957. $P$ is a square block matrix with square unit blocks of orders $m_1, m_2, \ldots, m_n$ along the main diagonal, and the block lying at the intersection of the $i$th block row and the $j$th block column coincides with matrix $X$, while all other off-diagonal blocks are zero. Similarly, $Q$ is a block matrix with square unit blocks of orders $n_1, n_2, \ldots, n_n$ on the main diagonal, with matrix $Y$ at the intersection of the $j$th block row and the $i$th block column, and with zero blocks elsewhere.

955. Hint. From the second block row subtract the first multiplied on the left by the matrix $CA^{-1}$ and, using the preceding problem, show that the rank remains unchanged in the process.

959. Solution. The same series of elementary operations that carries matrix $R$ into $R_1$ carries the matrix $T = \begin{pmatrix} A & B \\ -C & -CA^{-1}B \end{pmatrix}$ into the matrix $T_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & X - CA^{-1}B \end{pmatrix}$. By the preceding problem, the rank of $T$ is equal to $n$. Since elementary operations do not alter the rank of a matrix, the rank of $T_1$ is equal to the rank of $T = n$ and the rank of $A_1$ is the rank of $A = n$. Therefore, $X - CA^{-1}B = 0$; $X = CA^{-1}B$.

960. $A^{-1} = \begin{pmatrix} -20 & 8 & 11 \\ 17 & -7 & -9 \\ -2 & 1 & 1 \end{pmatrix}$. 961. $x = 1, y = 2, z = 3$.

962. $X = \begin{pmatrix} 9 & 8 \\ 2 & 3 \end{pmatrix}$.
964. (a) For the right direct product, take the lexicographic arrangement of pairs: 
\((1, 1), (1, 2), \ldots, (1, n), (2, 1), (2, 2), \ldots, (m, 1), (1, 2), (2, 2), \ldots, (m, n)\). For the left product, take the arrangement \((1, 1), (2, 1), \ldots, (m, 1), (1, 2), (2, 2), \ldots, (m, 2), \ldots, (m, n)\) that is obtained by lexicographic recording of the same pairs read from right to left. The properties (b), (c), and (d) follow directly from the definition. The property (e) follows from the relations of the preceding problem and (d) of the present problem. The properties of the left product follow from the appropriate properties of the right product with the aid of relations (b).

965. Hint. Show that a change in the numbering of the pairs does not alter the determinant \(|A \times B|\) and, using property (c) of problem 963, express it in the form \(|A \times B| = |(AE_n) \times (EB_n)| = |A \times E_n| \times |E_n \times B|\).

966. Hint. Use the proposition, that from \(|A| = 0\) follows the rank of \(\bar{A} \leq 1\) that was proved in problem 959.

967. Solution. Set \(AB = C\) and denote the cofactors by \(A_{ij}, B_{ij}, C_{ij}\); respectively, and the minors of the elements of the \(i\) th row and the \(j\) th column of matrices \(A, B, C\) by \(M_{ij}, N_{ij}, P_{ij}\) respectively.

Then, applying the expression for the minor of a product of two matrices in terms of the minors of the matrices (problem 913), we obtain
\[\hat{c}_{ij} = c_{ij} = (-1)^{ij} P_{ij} = (-1)^{ij} \sum_{h=1}^{n} M_{jkh}N_{hi},\]
\[= \sum_{h=1}^{n} (-1)^{ij + h} M_{jkh}(-1)^{h+i} = \sum_{h=1}^{n} A_{jkh}N_{hi} = \sum_{h=1}^{n} B_{jih}A_{hi},\]
whence \(\hat{C} = \hat{B} \cdot \hat{A}\).

For the nonsingular matrices \(A\) and \(B\), the same result is obtained more readily thus: by the preceding problem \(\hat{A} = |A| \cdot A^{-1}\), whence \((\hat{A}B) = |A| \cdot (AB)^{-1} = |A| \cdot |B| \cdot |B^{-1}A^{-1}| = B \cdot A\).

968. Solution. By virtue of problem 956, from \(AB = E_n\) follows \(A_pB_p = E_n\), where \(N = C_p\), whence
\[\sum_{1 \leq j_1 < i_2 < \ldots < i_p \leq n} A \begin{pmatrix} j_1, & j_2, & \ldots, & j_p \\ i_1, & i_2, & \ldots, & i_p \end{pmatrix} B \begin{pmatrix} k_1, & k_2, & \ldots, & k_p \end{pmatrix}.
\]

969. Hint. Use problem 943.

970. For example, the numbering of the combinations is in lexicographic order in which the combination \(i_1 < i_2 < \ldots < i_p\) precedes the combination \(i_1' < i_2' < \ldots < i_p'\). If the first nonzero difference \(i_1 - i_1'\), \(i_2 - i_2'\), \ldots, \(i_p - i_p'\) is positive.

971. Hint. Prove the proposed equality first for a triangular matrix \(A\) by using the fact that changing the numbering of the combinations does not change the determinant of the associated matrix \(A_p\) and also by taking advantage of the preceding problem. Reduce the general case to triangular matrices with the aid of problems 928 and 969.

972. Solution. By virtue of problem 956, from \(AB = E\) follows \(A_pB_p = E_N\), where \(N = C_p\), whence
\[\sum_{1 \leq j_1 < i_2 < \ldots < i_p \leq n} A \begin{pmatrix} j_1, & j_2, & \ldots, & j_p \\ i_1, & i_2, & \ldots, & i_p \end{pmatrix} B \begin{pmatrix} k_1, & k_2, & \ldots, & k_p \end{pmatrix}.
\]
\[= \begin{cases} 1, & \text{if } \sum_{p=1}^{p} (j_p - k_p)^2 = 0, \\ 0, & \text{if } \sum_{p=1}^{p} (j_p - k_p)^2 > 0. \end{cases}\]

973. Hint. Use the Laplace theorem and problem 903.

974. Solution. By virtue of problem 956, from \(AB = E_n\) follows \(A_pB_p = E_N\), where \(N = C_p\), whence
\[\sum_{1 \leq j_1 < i_2 < \ldots < i_p \leq n} A \begin{pmatrix} j_1, & j_2, & \ldots, & j_p \\ i_1, & i_2, & \ldots, & i_p \end{pmatrix} B \begin{pmatrix} k_1, & k_2, & \ldots, & k_p \end{pmatrix}.
\]
\[= \begin{cases} |A|, & \text{if } \sum_{p=1}^{p} (j_p - k_p)^2 = 0, \\ 0, & \text{if } \sum_{p=1}^{p} (j_p - k_p)^2 > 0, \end{cases}\]

where \(i_1 < i_2 < \ldots < i_p\), together with \(i_1 < i_1' < \ldots < i_p\) and \(k_1 < k_1' < \ldots < k_p\), constitute a complete system of indices \(1, 2, \ldots, n\).

Since the system of linear equations with the nonsingular matrix \(A\) for the given constant terms has a unique solution and since the right members of equation (2) differ from the right members of equation (2) in the factor \(|A|\) alone, then so also should the left-hand members differ in that factor, whence follow the required equations (1).

975. Hint. Use the Laplace theorem and problem 903.
984. **Hint.** Prove that the polynomials $D_k(\lambda)\$ do not change under elementary operations and that in the case of a normal diagonal form, $D_k(\lambda) = \prod_{k=1}^{n} E_k(\lambda) \ (k = 1, 2, \ldots, n)$.

985. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda (\lambda - 1) (\lambda - 2) & 0 & 0 \\
0 & 0 & \lambda (\lambda - 1) (\lambda - 2) & 0 \\
0 & 0 & 0 & \lambda (\lambda - 1) (\lambda - 2)
\end{pmatrix}
\]

986. 
\[
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda (\lambda - 1) (\lambda - 2) & 0 \\
0 & 0 & 0 & \lambda (\lambda - 1) (\lambda - 2)
\end{pmatrix}
\]

987. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
p & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & 0 & p
\end{pmatrix}
\]

where $p = (\lambda - 1) (\lambda - 2) (\lambda - 3) (\lambda - 4)$.

988. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
p & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & 0 & p^2
\end{pmatrix}
\]

where $p$ is the product of polynomials $a, b, c, d$ divided by the product of the leading coefficients of the polynomials.

989. 
\[
\begin{pmatrix}
d(\lambda) & 0 & 0 & 0 \\
0 & f(\lambda) g(\lambda) & 0 & 0 \\
0 & 0 & c(\lambda) & 0 \\
0 & 0 & 0 & d(\lambda)
\end{pmatrix}
\]

where $d(\lambda)$ is the largest common divisor of the polynomials $f(\lambda)$ and $g(\lambda)$ having leading coefficient equal to unity and having $c$ as the product of the leading coefficients of the polynomials.

990. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & fgh & 0 & 0 \\
0 & 0 & fgh
\end{pmatrix}
\]

where $a$, $b$, $c$ are respectively the largest common divisors of $x$ and $y$ and $z$, $f$ and $k$, $f$ and $k$, and $f$ and $g$, the leading coefficients of all polynomials $a$, $b$, $c$, $d$ being unity.

993. 
\[
\frac{abc}{d} \begin{pmatrix}
\lambda^4 & 0 & 0 & 0 \\
0 & \lambda^3 & 0 & 0 \\
0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix}
\]

where $d$ is the largest common divisor of $f$, $g$ and $h$; and $a$, $b$, $c$ are respectively the largest common divisors of $p$, $q$, $f$, $h$, $f$ and $k$, and $f$ and $g$, the leading coefficients of all polynomials $a$, $b$, $c$, $d$ being unity.

994. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

995. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & f(\lambda)
\end{pmatrix}
\]

where $f(\lambda) = \lambda^5 + 5\lambda^4 + 4\lambda^3 + 3\lambda^2 + 2\lambda + 1$.

996. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

997. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

where $\beta \neq 0$, $\alpha$

998. 
\[
\begin{pmatrix}
\lambda - 1 & 0 & 0 & 0 \\
0 & \lambda - 1 & 0 & 0 \\
0 & 0 & \lambda - 1 & 0 \\
0 & 0 & 0 & \lambda - 1
\end{pmatrix}
\]

999. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\lambda - 1)^3
\end{pmatrix}
\]

when $n$ is the order of the given matrix.
Equivalent. 1001. They are not equivalent.

1002. The matrices $A$ and $C$ are equivalent, but they are not equivalent to matrix $B$. 1003. The unit matrix.

1005. Hint. Take advantage of the fact that an elementary transformation of the rows of matrix $A$ reduces to a premultiplication of the columns by a special unimodular $\lambda$-matrix. Furthermore, if $B = P_1 P_2 \ldots P_m A Q_1 Q_2 \ldots Q_n$ where $P_i, Q_i$ are special unimodular $\lambda$-matrices, then set $P = p_1^{-1} \ldots p_{m-1}^{-1}$ and $Q = Q_n Q_{n-1} \ldots Q_1$. When proving sufficiency, make use of the answer to problem 1003.

1006. $B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 - 1 \end{pmatrix}$; $P = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

1007. $B = \begin{pmatrix} \lambda + 2 & 0 \\ 0 & \lambda^2 - 2 \lambda + 4 + 1 \end{pmatrix}$; $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1008. $B = \begin{pmatrix} 0 & 0 \\ 0 & \lambda^2 - 1 \end{pmatrix}$; $P = \begin{pmatrix} -2 \lambda - 3 & 2 \lambda + 4 \\ 1 & 0 \end{pmatrix}$

1009. For example,

$$P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

1010. For example,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1011. For example,

$$P = \begin{pmatrix} 1 & 0 \\ 11 \lambda + 8 & 2 \lambda - 1 \\ 0 & -1 \\ -2 \lambda - 1 & 2 \end{pmatrix}$$

1012. For example,

$$P = \begin{pmatrix} -3 & -1 \\ -1 & 0 \end{pmatrix}$$

1013. For example,

$$P = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$

1014. For example,

$$P = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

1015. $E_1 (A) = 0$; $E_2 (A) = \lambda - 1$; $E_3 (A) = (\lambda - 1) (\lambda^2 - 1)$.

1016. $E_1 (A) = \lambda + 1$; $E_2 (A) = \lambda^2 - 1$; $E_3 (A) = \lambda^3 - \lambda$.

1017. $E_1 (A) = \lambda^2 + 1$; $E_2 (A) = \lambda^4 - \lambda^2 + \lambda - 1$; $E_3 (A) = 0$.

1018. $E_1 (A) = 1$; $E_2 (A) = \lambda^2 - \lambda + 1$; $E_3 (A) = \lambda^3 + 1$.

1019. $E_1 (A) = 0$; $E_2 (A) = 1$; $E_3 (A) = \lambda + 1$.

1020. $E_1 (A) = 0$; $E_2 (A) = \lambda - \alpha$ if $\beta \neq 0$, $E_1 (A) = 0$; $E_2 (A) = \lambda - \alpha$; $E_3 (A) = 0$.

1021. $\lambda + 1$, $\lambda - 1$.

1022. $\lambda + 1$, $\lambda - 1$.

1023. There are no elementary divisors.

1024. $\lambda + 1$, $\lambda - 1$, $\lambda + 2$, $\lambda + 2$, $\lambda + 2$.

1025. There are no elementary divisors.

1026. In the field of rational numbers: $\lambda^2 + 1$, $\lambda^2 - 3$; in the field of real numbers: $\lambda^2 + 1$, $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$; in the field of complex numbers: $\lambda + i$, $\lambda - i$, $\lambda + i\sqrt{3}$.

1027. The field of rational numbers: $\lambda^2 + 1$, $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$.

1028. In the field of real numbers: $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$, $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$.

1029. In the field of complex numbers: $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$, $\lambda + \sqrt{3}$, $\lambda - \sqrt{3}$.

1030. In the field of complex numbers: 

$$\left(\lambda - 1, \lambda + 1\right)^2, \left(\lambda + 1\right)^2, \left(\lambda - 1\right)^2, \left(\lambda + 1\right)^2, \left(\lambda - 1\right)^2$$.
1029. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda + 1 & 0 & 0 \\
0 & 0 & \lambda^2 + 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

1030. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda + 2 & 0 & 0 \\
0 & 0 & \lambda^2 + 2\lambda - 4 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

1031. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda - 1 & 0 & 0 \\
0 & 0 & \lambda^2 + \lambda - 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

1032. Hint. Suppose \( e(\lambda) \) is some kind of irreducible factor in the decomposition of at least one diagonal element, let \( s \) be the number of diagonal elements different from zero, and let \( 0 < \alpha_1 < \alpha_2 < \ldots \) be the collection of exponents which \( e(\lambda) \) encounters in these elements. Show that for \( k = 1, 2, \ldots, s \) the divisor of the minors \( D_k(\lambda) \) is exactly divisible by \( [e(\lambda)]^{\alpha_1 + \alpha_2 + \ldots + \alpha_k} \) and the invariant factor \( F_k(\lambda) \) is exactly divisible by \( [e(\lambda)]^{\alpha_k} \).

1033. Hint. Use elementary operations to bring each diagonal block to diagonal (for example, normal) form and take advantage of the preceding problem.

1034. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda (\lambda + 1) & 0 & 0 \\
0 & 0 & \lambda (\lambda + 1) (\lambda - 1) & 0 \\
0 & 0 & 0 & \lambda^2 (\lambda - 1)^2 (\lambda - 1)^2
\end{pmatrix}
\]

1035. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda^3 - 4\lambda & 0 & 0 \\
0 & 0 & \lambda^3 - 4\lambda & 0 \\
0 & 0 & 0 & \lambda^4 - 4\lambda^2
\end{pmatrix}
\]

1036. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & f(\lambda) & 0 & 0 \\
0 & 0 & f(\lambda) & 0 \\
0 & 0 & 0 & [f(\lambda)]^2
\end{pmatrix}, \text{ where } f(\lambda) = \lambda^2 + \lambda^3 - 6\lambda.
\]

1037. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda^2 - 1 & 0 & 0 \\
0 & 0 & (\lambda^2 - 1)^2 & 0 \\
0 & 0 & 0 & (\lambda^2 - 1)^2
\end{pmatrix}
\]

1038. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda - 1 & 0 & 0 \\
0 & 0 & \lambda^2 - 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

1039. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \lambda^2 - 4 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

1040. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \lambda^2 + \lambda - 6 & 0 \\
0 & 0 & 0 & \lambda^2 + \lambda - 6
\end{pmatrix}
\]

1041. 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda - 1 & 0 & 0 \\
0 & 0 & \lambda^4 - 1 & 0 \\
0 & 0 & 0 & \lambda^4 - 2\lambda^2 + 1
\end{pmatrix}
\]

1042. \( D_1 = 1, D_2 = 2, D_3 = 4, D_4 = 320. \)

1043. \( D_1 = 3, D_2 = 18, D_3 = 324, D_4 = 11664. \)

1044. Hint. Make use of the preceding problem when proving the existence of representation of the given type. When proving uniqueness of the two representations of the given type \( A = P_1 R_1 = P_2 R_2 \), derive that the matrix \( C = P_2^t P_1 = R_1 R_2 \) is a unimodular and triangular \( \lambda \)-matrix whose elements on the main diagonal have leading coefficient unity, which means that they themselves are equal to unity. Then, equating the elements of the \( k \)th row in the equation \( C R_1 = R_2 \) and taking into account the condition for the exponents of the elements \( R_1 \) and \( R_2 \), show that all elements of matrix \( C \) to the right of the main diagonal are equal to zero, that is, \( C \) is a unit matrix. From this obtain the equalities \( P_1 = P_2 \) and \( R_1 = R_2 \).

1045. Hint. For example, \( A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}. \)

1046. Scalar matrices. Hint. Show that if \( AT = TA \) for any non-singular matrix \( T \), then it holds true for all matrices \( T \). To do this, represent the singular matrix \( T \) as \( T = (T - \alpha E) + \alpha E \) where \( \alpha \neq 0 \) and is chosen so that \( |T - \alpha E| \neq 0 \). Then apply problem 818. Note. This method of solution may prove inadequate for matrices with elements taken from a finite field where there may be an element \( \alpha \) for which \( \alpha^2 = 1 \), and where \( \alpha^2 \neq 1 \neq \alpha \). Then apply problem 818.

1047. For example, \( A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}. \)

1048. Scalar matrices. Hint. Show that if \( AT = TA \) for any non-singular matrix \( T \), then it holds true for all matrices \( T \). To do this, represent the singular matrix \( T \) as \( T = (T - \alpha E) + \alpha E \) where \( \alpha \neq 0 \) and is chosen so that \( |T - \alpha E| \neq 0 \). Then apply problem 818. Note. This method of solution may prove inadequate for matrices with elements taken from a finite field where there may be an element \( \alpha \) for which \( \alpha^2 = 1 \), and where \( \alpha^2 \neq 1 \neq \alpha \). Then apply problem 818. Note. This method of solution may prove inadequate for matrices with elements taken from a finite field where there may be an element \( \alpha \) for which \( \alpha^2 = 1 \), and where \( \alpha^2 \neq 1 \neq \alpha \). Then apply problem 818.
1049. For the matrix $T$, one can take the matrix obtained from the unit matrix by interchanging the $i$th and $j$th rows.

1050. Hint. Take advantage of the preceding problem.

1058. $A = (B - \lambda E) = \begin{pmatrix} 2A + 1 & 1 & 2 \\ A + 2 & 1 \\ 2A + 1 \end{pmatrix}.$

1059. $A = \lambda^2 \lambda - 2 + 6 - 32 + 7 - 5 + 4.$


1061. Solution. Suppose $P = (B - \lambda E) P_1 + P_0$ and $Q = Q_1 (B - \lambda E) + Q_0$. Using these relations, reduce

$$B - \lambda E = P (A - \lambda E) Q,$$

by the form

$$B - \lambda E = P_0 (A - \lambda E) Q_0 = P (A - \lambda E) Q_1 (B - \lambda E) + (B - \lambda E) P_1 (A - \lambda E) Q_1 (B - \lambda E).$$

Substituting $P (A - \lambda E) = (B - \lambda E) Q_1$ and $(A - \lambda E) Q = P^{-1} (B - \lambda E)$ into this equation on the basis of (1) we get

$$(B - \lambda E) = P_0 (A - \lambda E) Q_0 = (B - \lambda E) P_1 (A - \lambda E) Q_1 (B - \lambda E).$$

The expression in square brackets in the right-hand member of the equation must be zero because otherwise the right-hand member would have a power in $\lambda$ not less than two, whereas the power of the left-hand member does not exceed unity. Therefore, $B - \lambda E = P_0 (A - \lambda E) Q_0$. Equating the coefficients of $\lambda$ and the constant terms in this equation, we get: $P_0 Q = E$ and $B = P_0 A Q_0$.

1063. Similar. 1064. Similar. 1065. The matrices $A$ and $C$ are similar, but they are not similar to matrix $B$.

1066. The matrices $B$ and $C$ are similar, but they are not similar to matrix $A$.

1067. For example, $T = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$. Hint. To obtain the simplest possible answer, strive towards the simplest elementary operations on the columns of the matrices $A - \lambda E$ and $B - \lambda E$.

1068. For example, $T = \begin{pmatrix} 0 & 2 \\ -1 & 10 \end{pmatrix}$. Hint. To reduce the matrix $A - \lambda E$ to normal diagonal form, from the second row multiply by $6$ subtract the first row multiplied by $\lambda + 16$, and from $6$ times the first column subtract the second column multiplied by $\lambda - 17$.

1069. For example,$T = \begin{pmatrix} 1 & -3 & 3 \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$

1070. Hint. Obtain $\alpha_0$ as a sum of all factors in the determinant $|A - \lambda E|$ that appear in products of $k$ elements each of the main diagonal and taken when $\lambda = 0$.

1071. $\lambda_k = \alpha_k + \alpha_{k-1} + \ldots + \alpha_1$, $\lambda_0 = \ldots = \lambda_n = 0$. Hint. Apply the preceding problem.

1074. Hint. Apply problem 1070 to the matrix $B = A - \lambda E$, and show that the characteristic polynomial $|B - \mu E|$ of matrix $B$, after the substitution $\mu = \lambda - \lambda_0$, passes into the characteristic polynomial $|A - \lambda E|$ of matrix $A$.

1075. For a triangular matrix of the form

$$A = \begin{pmatrix} \lambda_0 a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & \lambda_0 a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_0 a_{nn} \end{pmatrix},$$

where $a_{ij} \neq 0$ ($i = 1, 2, \ldots, n - 1$), we have $d = 1$. For a diagonal matrix of order $n$ in which $p$ elements of the main diagonal are equal to $\lambda_0$ we have $d = p$.

1076. Hint. Prove that

$$|A^{-1} - \lambda E| = (-\lambda)^n |A^{-1}| |A - \lambda E|.$$

1077. Hint. Multiply together the equalities

$$|A - \lambda E| = (\lambda - \lambda_1) (\lambda - \lambda_2) \ldots (\lambda - \lambda_n),$$

$$A + \lambda E = (\lambda + \lambda_1) (\lambda + \lambda_2) \ldots (\lambda + \lambda_n)$$

and replace $\lambda^p$ by $\lambda$.

1078. Hint. Multiply together the equality $|A_E - \lambda E| = (\lambda - \lambda_1) \times (\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$ and all equalities obtained from it by replacing $\lambda$ by $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, and $\lambda$, and in the resulting equality substitute $\lambda$ for $\lambda$. 

1079. Solution. Let $f(\lambda) = a_0 \sum_{j=1}^{n} (\lambda - \mu_j)$ and, besides, $\varphi(\lambda) = a_0 \sum_{j=1}^{n} (\lambda - \mu_j)$. Setting $\lambda = A$ in $f(\lambda)$ we get: $f(A) = a_0 \sum_{j=1}^{n} (A - \mu_j B)$.

Passing from matrices to determinants, we obtain

$$|f(A)| = a_0^n \sum_{j=1}^{n} |A - \mu_j E| = a_0^n \sum_{j=1}^{n} \varphi(\mu_j) = a_0^n \sum_{j=1}^{n} \sum_{i=1}^{n} (\lambda_i - \mu_j) =$$

$$= \sum_{i=1}^{n} a_0 \sum_{j=1}^{n} (\lambda_i - \mu_j) = \sum_{i=1}^{n} f(\lambda_i).$$

On the other hand, $|f(A)| = a_0^n \sum_{j=1}^{n} \varphi(\mu_j) = R(f, \varphi)$. 

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1080. Hint. Apply the equality of the preceding problem to the polynomial \( g(x) = f(x) - \lambda \), where \( \lambda \) is an arbitrary number.

1081. Hint. Apply the equality \( \| f(A) \| = \| g(A) \| \) and make use of problem 1079 and 1080.

1082. Hint. If at least one of the matrices \( A \) and \( B \) is nonsingular, then the assertion follows from the similarity of the matrices \( AB \) and \( AB \) (see problem 1047). In the general case, problems 930 and 1070 may be used. For matrices over a field with an infinite (or sufficiently large) number of elements, fulfillment of the required equality for nonsingular matrices implies that it is identically fulfilled. Finally, for matrices with numerical elements, the equality for a singular matrix \( A \) may be obtained by a passage to the limit. For example, if \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of the singular matrix \( A \), then we take a sequence of numbers \( \epsilon_k, \epsilon_k^2, \ldots \) such that all of them are different from \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( \lim \epsilon_k = 0 \). The matrix \( A_k = A - \epsilon_k E \) is nonsingular. Hence, \( \| A_k B - \lambda E \| = \| B A_k - \lambda E \| \).

Passing to the limit as \( k \to \infty \), we obtain the needed equality.

1083. The eigenvalues (with account taken of multiplicity) are: \( \lambda_k = f(\epsilon_k) \), where \( f(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1} \) and \( a_k = 2\pi k + i \sin \frac{2\pi k}{n} \) for \( k = 0, 1, 2, \ldots, n - 1 \). Hint. Apply the expression for the circulant of problem 479 to the circulant \( A - \lambda E \), where \( \lambda \) is a parameter.

1084. Solution. Applying problem 304 to the determinant \( | A - \lambda E | \), where \( \lambda \) is the eigenvalue, set \( \alpha + \beta = \lambda \), \( \alpha \beta = -1 \). Then \( | A - \lambda E | = \alpha^n + \alpha^{n-1} \beta + \ldots + \beta^n \). From \( \alpha \beta = -1 \) we find \( \alpha \neq 0 \) and \( \beta \neq 0 \). Furthermore, \( \alpha \neq \beta \), since from \( \alpha = \beta \) and \( | A - \lambda E | = 0 \) it should follow that \( \alpha = 0 \). Therefore, \( | A - \lambda E | = \frac{\alpha^n + \beta^n}{\alpha - \beta} = 0 \), whence \( \left( \frac{\alpha}{\beta} \right)^{n+1} - 1; \frac{\alpha}{\beta} = \frac{\cos \frac{2\pi k}{n+1} + i \sin \frac{2\pi k}{n+1}}{2 k} \frac{\alpha}{\beta} = 1 \). Solving this equation simultaneously with \( \alpha \beta = -1 \), we find \( \alpha = \pm i \left( \cos \frac{\pi k}{n+1} + i \sin \frac{\pi k}{n+1} \right) \), \( \beta = = \mp i \left( \cos \frac{\pi k}{n+1} - i \sin \frac{\pi k}{n+1} \right) \). Here the signs \( \pm \) must be taken coincident for \( \alpha \) and \( \beta \), since \( \alpha \beta = -1 \), whence \( \lambda = - (\alpha + \beta) = = \mp 2 \pi \cos \frac{\pi k}{n+1} \) for \( k = 1, 2, \ldots, n \). All these must be eigenvalues.

But among them there are some that are equal since \( \cos \frac{\pi k}{n+1} = = - \cos \left( \frac{\pi k}{n+1} \right) \frac{n+1}{n+1} \). All distinct eigenvalues are contained in the system

\[ \lambda_k = 2 \pi \cos \frac{\pi k}{n+1} \quad (k = 1, 2, \ldots, n). \]

But the degree of the characteristic polynomial is equal to \( n \). Thus the latter system contains all eigenvalues, and there are no multiple roots.

1085. Hint. Prove that the Jordan submatrix of order \( k \) with the number \( \alpha \) on the diagonal has a unique elementary divisor \( \lambda - \alpha E \).

Construct the Jordan matrix \( A \) whose Jordan submatrices are related as indicated with the elementary divisors of the matrix \( A - \lambda E \) and, using problems 1035 and 1081, prove that the matrices \( A \) and \( A \) and, using the coincidence of the elementary divisors of the matrices \( B - \lambda E \) and \( C - \lambda E \), again apply problem 1035 to make sure that the matrices \( B \) and \( C \) coincide to within the order of the blocks.

1086. \[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

1087. \[ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

1088. The problem has not been posed properly. No such invariant factors are possible in a matrix \( A - \lambda E \) of the fourth order.

1089. \[ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \]

1090. \[ \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \]

1091. \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \]

1092. \[ \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \]

1093. \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

1094. \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

1095. \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]

1096. \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

1100. \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
1103. \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 + i & 0 \\
0 & 0 & 2 - i
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 2 & 1
\end{pmatrix}, \quad \text{where} \quad e = -\frac{1}{2} \pm \frac{i}{2} \sqrt{3}
\]
is one of the complex values of \( \sqrt[3]{i} \).

1104. \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 + 3i & 0 \\
0 & 0 & 2 - 3i
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

1106. \[
\begin{pmatrix}
2 & 1 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

1108. \[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

1110. The same as in problem 1109. 1111. The same as in problem 1109.

1112. \[
\begin{pmatrix}
\alpha_0 & 0 & 0 & \cdots & 0 \\
0 & \alpha_1 & 0 & \cdots & 0 \\
0 & 0 & \alpha_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \alpha_{n-1}
\end{pmatrix}, \quad \text{where} \quad \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \text{ are all}
\]
the values of \( \sqrt[n]{\alpha} \), that is, \( \alpha_k = \alpha \left( \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n} \right) \) \((k = 0, 1, 2, \ldots, n-1)\).

1115. One Jordan submatrix with the number \( \alpha \) on the main diagonal.

1116. In the field of rational numbers it is similar to the matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]

1119. In the field of real numbers it is similar to the matrix
\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & \sqrt{3} & 0 \\
0 & 0 & -\sqrt{3}
\end{pmatrix}
\]

1120. In the field of complex numbers it is similar to the matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 + 3i & 0 \\
0 & 0 & 2 - 3i
\end{pmatrix}
\]

1121. It is not similar to the diagonal matrix in any field.

1122. The diagonal matrix with elements \( \pm i \) on the main diagonal.

Note. The assertion does not hold true for matrices over a field of characteristic 2. For example, \( \begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix} \quad E \) and matrix \( \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \) cannot be reduced to diagonal form due to the uniqueness of the Jordan form.

1123. If \( n \) is the period of the matrix \( A \), that is, the smallest of the natural numbers \( k \) for which \( A^k = E \), then on the main diagonal the diagonal matrix has certain of the \( n \) values of the root \( \sqrt[n]{i} \).

Note. The result is not correct; for matrices over fields of a finite characteristic, for instance, for \( \mathbb{F} \) matrix of order \( \leq p \) over a field of characteristic \( p \),
\[
A = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]
and the equality \( AP = E \) holds true.

1126. For the scalar matrices \( A = \alpha I \) and for them alone. There is only one such matrix for the given order \( n \).

1127. For example, for the matrices
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

1128. (a) \( \lambda = 1 \). (b) \( \lambda = 0 \).

1129. For the field of real numbers it is similar to the matrix
\[
\begin{pmatrix}
\lambda^2 - 4\lambda + 4 & 0 \\
0 & \lambda^2 - 5\lambda + 6
\end{pmatrix}
\]

1130. For the matrices
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[\varphi(\lambda) = (\lambda - 1)^4, \psi(\lambda) = (\lambda - 1)^6\] but these matrices are not similar due to the uniqueness of the Jordan form of any matrix.

1131. \[
\begin{pmatrix}
\alpha^k & C_k^0 \alpha^{k-1} & C_k^1 \alpha^{k-2} & \cdots & C_k^{k-2} \alpha \alpha^{n-3} & C_k^{k-1} \alpha^{n-4} \\
0 & \alpha^k & C_k^1 \alpha^{k-1} & \cdots & C_k^{k-3} \alpha \alpha^{n-2} & \cdots \\
0 & 0 & \alpha^k & \cdots & \alpha \alpha^{n-1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \alpha^k
\end{pmatrix}
\]

for \( k \leq n - 1 \) it is necessary to put \( C_k^0 = 1 \), and \( C_k^i = 0 \) for \( k > n \).
1138. Hint. Set \( A = a + i \) in the equality

\[ f(x) = f(x) + \frac{f''(x)}{2!} (x-\alpha)^2 + \ldots + \frac{f^{(n)}(x)}{n!} (x-\alpha)^n \]

where \( x = \alpha \).

1140. One Jordan submatrix with the number \( a^2 \) on the diagonal.

1141. If \( n > 1 \) is the order of the Jordan submatrix \( A \) with zero on the diagonal, then the Jordan form of the matrix \( A^n \) consists of \( n \) blocks with zero on the diagonal, the blocks having the order \( \frac{n}{2} \) for even \( n \) and the orders \( \frac{n+1}{2} \) for odd \( n \).

Hint. Use problem 1130 to find the minimal polynomials of the matrices \( A \) and \( A^2 \) and show that the blocks of the Jordan form of the matrix \( A^n \) have orders that do not exceed \( \frac{n}{2} \) for even \( n \) and \( \frac{n+1}{2} \) for odd \( n \).

1142. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of the matrix \( A \) (with account taken of multiplicity), then the eigenvalues of the matrix \( A^2 \) (also with account taken of multiplicity) are equal to the values \( \lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2 \). This is to say, to all possible products made up of \( p \) eigenvalues of \( A \).

Hint. Establish the fact that a change in the numbering of the combinations of \( p \) numbers selected from the \( n \) numbers 1, 2, \ldots, \( n \) the matrix \( A^2 \) becomes a similar matrix, then use problem 970, pass to the Jordan form \( A^2 \), and apply the properties of associated matrices of problem 963.

1146. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of \( A \) and \( \mu_1, \mu_2, \ldots, \mu_n \) are eigenvalues of \( B \), then the eigenvalues of \( AB \) are equal to \( \lambda_1 \mu_1, \lambda_2 \mu_2, \ldots, \lambda_n \mu_n \). Solution. Suppose the Kronecker product \( A \times B \) is determined by the arrangement of \( \lambda_1, \lambda_2, \ldots, \lambda_p \), \( \mu_1, \mu_2, \ldots, \mu_q \) pairs of numbers (i, j) (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q). The transposition of \( \lambda_i \) and \( \mu_j \) produces in the matrix \( A \times B \) an interchange of the \( i \)th and \( j \)th rows and the \( i \)th and \( j \)th columns and, hence, will carry matrix into a similar matrix (problem 1049). Since any permutation reduces to a series of transpositions, the eigenvalues of all Kronecker products of the form \( A \times B \) coincide and one can, for example, consider the right direct product \( A \times B \) (problem 964). Let \( A \) be equal to \( C \times D \), and \( B = -B_i \), where \( A_i \) and \( B_j \) are Jordan matrices. Applying the property (c) of problem 963, we find: \( A \times B = C \times D \), therefore the matrices \( A \times B \) and \( B_i \) are similar and their eigenvalues coincide. But \( A_i \times B_j \) is a triangular matrix with elements \( \lambda_i \mu_j \), \( (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q) \) on the main diagonal, and this completes the proof of our assertion.

1147. Hint. Show that \( g(A) = h(A) \) if and only if \( g(\lambda) = h(\lambda) \) is divisible by \( \psi(\lambda) \).

1149. In the given case the minimal polynomial coincides with the characteristic polynomial (up to sign). \( r(\lambda) \) is an ordinary Lagrange interpolation polynomial,

\[ r(\lambda) = \sum_{\lambda} \prod_{\lambda \neq \lambda_i} (\lambda_i - \lambda) \]

where \( \lambda \) are the eigenvalues of matrix \( A \) (by hypothesis, distinct).

1150. \( r(\lambda) \) is an ordinary Lagrange interpolation polynomial.

\[ r(\lambda) = \sum_{\lambda} \prod_{\lambda \neq \lambda_i} (\lambda_i - \lambda) \]

where \( \lambda \) are the eigenvalues of matrix \( A \) (by hypothesis, distinct).
Solution. First, we show that if the Lagrange-Sylvester interpolation polynomial \( r(\lambda) \) exists, then it is determined by the equalities (1) and (2). Let

\[
\frac{r(\lambda)}{\psi(\lambda)} = \sum_{k=1}^{n} \left[ \frac{\alpha_k}{(\lambda - \lambda_k)^{r_k}} + \cdots + \frac{\alpha_k}{(\lambda - \lambda_k)^{r_k-1}} \right]
\]

be an expansion in partial fractions. Multiplying this equality by \( \psi(\lambda) \), we obtain (1). To establish (2), let us multiply (3) by \( (\lambda - \lambda_k)^{r_k} \) to get

\[
\frac{r(\lambda)}{\psi_k(\lambda)} = \alpha_{h,1} + \cdots + \alpha_{h,n} (\lambda - \lambda_k)^{-1} + (\lambda - \lambda_k)^{r_k} \psi_k(\lambda),
\]

where \( \psi(\lambda) \) is a rational function that is meaningful when \( \lambda = \lambda_k \) together with all its derivatives. Taking the \( (j-1) \) derivative of both sides of (4) for \( \lambda = \lambda_k \) and using the fact that the values of \( r(\lambda) \) and \( f(\lambda) \) on the spectrum of matrix \( A \) coincide, we thus obtain (2). Secondly, we show that the polynomial \( r(\lambda) \) defined by (1) and (2) is a Lagrange-Sylvester interpolation polynomial for the function \( f(\lambda) \) on the spectrum of matrix \( A \). From (1) it is evident that the degree of \( r(\lambda) \) is lower than that of \( \psi(\lambda) \). Furthermore, we set

\[
\psi_k(\lambda) = \frac{1}{\alpha_{h,1} + \cdots + \alpha_{h,n} (\lambda - \lambda_k) + \cdots + \alpha_{h,n} (\lambda - \lambda_k)^{-1} + (\lambda - \lambda_k)^{r_k} \psi_k(\lambda)}
\]

From the equalities (2) it follows that for \( \lambda = \lambda_k \) the values of the function \( q_k(\lambda) \) and its derivatives of order \( j < r_k \) coincide respectively with the values of the function \( f(\lambda) \) and of its derivatives of the same order. Therefore, setting \( \lambda = \lambda_k \) in \( r(\lambda) = \sum_{k=1}^{n} \frac{r(\lambda)}{\psi_k(\lambda)} \psi_k(\lambda) \) and the equalities obtained from it by \( j \)-fold differentiation \( (j < r_k) \), we obtain

\[
\frac{\partial^j}{\partial \lambda^j} r(\lambda) = \frac{\partial^j}{\partial \lambda^j} \psi_k(\lambda) \quad (j = 0, 1, \ldots, r_k - 1; k = 1, 2, \ldots, n),
\]

which is to say the values of \( r(\lambda) \) and \( f(\lambda) \) coincide on the spectrum of the matrix.

Hint. Show that the values of the Lagrange-Sylvester interpolation polynomial \( r(\lambda) \) for \( f(\lambda) \) coincide on the spectrum of matrix \( A \) with the values of \( f(\lambda) \) on the spectrum of each block of \( A_k \), and apply problem 1147.

\[
\frac{f(\lambda)}{\psi_k(\lambda)} = \frac{1}{\alpha_{h,1} + \cdots + \alpha_{h,n} (\lambda - \lambda_k) + \cdots + \alpha_{h,n} (\lambda - \lambda_k)^{-1} + (\lambda - \lambda_k)^{r_k} \psi_k(\lambda)}
\]

where \( \psi(\lambda) \) is a rational function that is meaningful when \( \lambda = \lambda_k \) together with all its derivatives. Taking the \( (j-1) \) derivative of both sides of (4) for \( \lambda = \lambda_k \) and using the fact that the values of \( r(\lambda) \) and \( f(\lambda) \) on the spectrum of matrix \( A \) coincide, we thus obtain (2). Secondly, we show that the polynomial \( r(\lambda) \) defined by (1) and (2) is a Lagrange-Sylvester interpolation polynomial for the function \( f(\lambda) \) on the spectrum of matrix \( A \). From (1) it is evident that the degree of \( r(\lambda) \) is lower than that of \( \psi(\lambda) \). Furthermore, we set

\[
\psi_k(\lambda) = \frac{1}{\alpha_{h,1} + \cdots + \alpha_{h,n} (\lambda - \lambda_k) + \cdots + \alpha_{h,n} (\lambda - \lambda_k)^{-1} + (\lambda - \lambda_k)^{r_k} \psi_k(\lambda)}
\]

From the equalities (2) it follows that for \( \lambda = \lambda_k \) the values of the function \( q_k(\lambda) \) and its derivatives of order \( j < r_k \) coincide respectively with the values of the function \( f(\lambda) \) and of its derivatives of the same order. Therefore, setting \( \lambda = \lambda_k \) in \( r(\lambda) = \sum_{k=1}^{n} \frac{r(\lambda)}{\psi_k(\lambda)} \psi_k(\lambda) \) and the equalities obtained from it by \( j \)-fold differentiation \( (j < r_k) \), we obtain

\[
\frac{\partial^j}{\partial \lambda^j} r(\lambda) = \frac{\partial^j}{\partial \lambda^j} \psi_k(\lambda) \quad (j = 0, 1, \ldots, r_k - 1; k = 1, 2, \ldots, n),
\]

which is to say the values of \( r(\lambda) \) and \( f(\lambda) \) coincide on the spectrum of the matrix.

Hint. Show that the values of the Lagrange-Sylvester interpolation polynomial \( r(\lambda) \) for \( f(\lambda) \) coincide on the spectrum of matrix \( A \) with the values of \( f(\lambda) \) on the spectrum of each block of \( A_k \), and apply problem 1147.

\[
\frac{f(\lambda)}{\psi_k(\lambda)} = \frac{1}{\alpha_{h,1} + \cdots + \alpha_{h,n} (\lambda - \lambda_k) + \cdots + \alpha_{h,n} (\lambda - \lambda_k)^{-1} + (\lambda - \lambda_k)^{r_k} \psi_k(\lambda)}
\]
The general solution is of the form
\[ \begin{pmatrix} 3 + 2\pi m \\ -5 + 2\pi n \\ 3 \\ -5 \\ 2 + 2\pi n \end{pmatrix}, \]
where \( i = \sqrt{-1} \) and \( n \) is any integer.

1170. \( \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \).


1173. \( e^A I = x^2 \), where \( x = a_{11} + a_{22} + \cdots + a_{nn} \) is the trace of matrix \( A \).

1174. Hint. Use problem 1161.

1175. \[ y_1^2 + y_2^2 - y_3^2 = y_1 \left( y_1 + y_2 - y_3 \right) \]
\[ x_1 = y_1 - \frac{5}{3} y_3 - \frac{1}{3} y_2, \quad x_2 = \frac{1}{3} y_3. \]

1176. \[ y_1^2 - y_2^2 - y_3^2 = \frac{1}{2} y_1 + \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1177. \[ y_1^2 - y_2^2 - y_3^2 = \frac{1}{2} y_1 - \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 + y_2 - y_3, \quad x_2 = y_3. \]

1178. \[ y_1^2 + y_2^2 - y_3^2 = y_1 \left( y_1 + y_2 + y_3 \right) \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1179. \[ y_1^2 - y_2^2 - y_3^2 = \frac{1}{2} y_1 + \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1180. \[ y_1^2 + y_2^2 - y_3^2 = y_1 \left( y_1 + y_2 + y_3 \right) \]
\[ x_1 = y_1 + y_2 + y_3, \quad x_2 = y_3. \]

1181. \[ y_1^2 + y_2^2 - y_3^2 = \frac{1}{2} y_1 + \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1182. \[ y_1^2 - y_2^2 - y_3^2 = \frac{1}{2} y_1 + \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 + y_2 - y_3, \quad x_2 = y_3. \]

1183. \[ y_1^2 - y_2^2 - y_3^2 = \frac{1}{2} y_1 + \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 + y_2 - y_3, \quad x_2 = y_3. \]

1184. \[ y_1^2 + y_2^2 - y_3^2 = y_1 \left( y_1 + y_2 + y_3 \right) \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1185. \[ y_1^2 - y_2^2 - y_3^2 = \frac{1}{2} y_1 + \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1186. \[ y_1^2 + y_2^2 - y_3^2 = y_1 \left( y_1 + y_2 + y_3 \right) \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1187. \[ y_1^2 + y_2^2 - y_3^2 = y_1 \left( y_1 + y_2 + y_3 \right) \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1188. \[ y_1^2 - y_2^2 - y_3^2 = \frac{1}{2} y_1 + \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1189. \[ y_1^2 - y_2^2 - y_3^2 = \frac{1}{2} y_1 + \frac{1}{3} y_2 \cdot \]
\[ x_1 = y_1 - y_2 - y_3, \quad x_2 = y_2 - y_3. \]

1190. \[ x_1 = y_3 - 3y_2 - 6y_1; \quad y_2 = x_1 + 3y_1; \quad y_3 = y_1. \]

1191. \[ x_2 = 2 \sqrt{2} y_1 + \sqrt{2} y_2 + 5 y_3; \quad x_2 = \frac{1}{2} \sqrt{2} y_1 + y_2 + 3 y_3. \]

1192. \[ x_3 = y_3; \quad y_3 = y_3 - \frac{1}{2} \sqrt{2} y_1 - y_2 - 3 y_3. \]

Hint. Reduce it to the preceding problem.

1193. \[ y_1^2 + y_2^2 + y_3^2 = \frac{3}{4} y_1 + \frac{4}{3} y_2 + \frac{5}{8} y_3 + \cdots + \frac{n + 1}{2^n} y_n; \]
\[ \sqrt{y_1} = x_1 + \frac{1}{2} \left( y_1 + y_2 + \cdots + y_n \right); \]
\[ y_2 - y_3 + \frac{1}{3} \left( y_2 + y_3 + \cdots + y_n \right); \]
\[ \cdots \cdots \]
\[ y_n = x_n. \]

1194. \[ y_1^2 + y_2^2 + y_3^2 = \frac{3}{4} y_1 + \frac{4}{3} y_2 + \frac{5}{8} y_3 + \cdots + \frac{n + 1}{2^n} y_n; \]
\[ \sqrt{y_1} = x_1 + \frac{1}{2} \left( y_1 + y_2 + \cdots + y_n \right); \]
\[ y_2 - y_3 + \frac{1}{3} \left( y_2 + y_3 + \cdots + y_n \right); \]
\[ \cdots \cdots \]
\[ y_n = x_n. \]

Hint. Reduce it to the preceding problem.

1195. \[ y_1^2 + y_2^2 - y_3^2 = \frac{3}{4} y_1 + \frac{4}{3} y_2 + \frac{5}{8} y_3 + \cdots + \frac{n + 1}{2^n} y_n; \]
\[ \sqrt{y_1} = x_1 + \frac{1}{2} \left( y_1 + y_2 + \cdots + y_n \right); \]
\[ y_2 - y_3 + \frac{1}{3} \left( y_2 + y_3 + \cdots + y_n \right); \]
\[ \cdots \cdots \]
\[ y_n = x_n. \]

Hint. Reduce it to the preceding problem.

1196. If \( n \) is even: \[ y_1^2 + y_2^2 + y_3^2 + \cdots + y_n^2; \]
\[ y_1 = \frac{1}{2} (x_1 + y_2) + \cdots + x_n + \cdots + x_n; \]
\[ y_2 = \frac{1}{2} (x_1 - x_2); \]
\[ y_3 = x_3 + \frac{1}{2} (x_1 + x_2 + \cdots + x_n); \]
\[ y_4 = x_4 + \frac{1}{2} (x_1 + x_2 + \cdots + x_n); \]
\[ \cdots \cdots \]
\[ y_n = x_n. \]
If \( n \) is odd: 
\[
y_i = x_1 + x_2 + \ldots + x_{n-2} + x_{n-1} + x_n; \\
y_i = \frac{x_1 + x_2 + \ldots + x_{n-2} + x_{n-1}}{2} \quad (i=1, 3, 5, \ldots, n-2); \\
y_i = x_{i-1} - x_i + x_{i+1} \quad (i=2, 4, 6, \ldots, n-1); \\
y_n = x_n. 
\]

1197. 
\[
\frac{n-1}{n} y_i^2 + \frac{n-2}{n} y_i^2 + \ldots + \frac{2}{n} y_i^2 + \frac{4}{n} y_i^2 + \frac{4}{n} y_i^2 + \ldots + \frac{1}{n} y_i^2; \\
y_1 = x_1 - x_2 + x_3 - \ldots + x_n; \\
y_2 = x_2 - x_3 + x_4 - \ldots + x_n; \\
\vdots \\
y_{n-1} = x_{n-1} - x_n; \\
y_n = x_n. 
\]

**Hint.** Represent the form as 
\[
f_i = \frac{n-1}{n} \sum_{i=1}^n x_i y_j \] 
and apply the method of induction. An alternative approach: perform the transformation 
\[
x_1 = x_1 - \epsilon; \\
x_2 = x_2 - \epsilon; \\
\vdots \\
x_{n-1} = x_{n-1} - \epsilon; \\
x_n = x_n; 
\]
then add the equalities and reduce the form 
\[
\sum_{i=1}^{n-1} (x_i - \epsilon)^2 + \sum_{i=1}^{n-1} (x_i - \epsilon)^2. 
\]

Examples. The form \( f_1 = -x_1^2 \) or the nondegenerate form \( f_2 = -x_1^2 + \ldots + 2x_1x_n \) has corner minors that are nonnegative, yet the forms themselves are not nonnegative. 

When proving sufficiency, expand \( D \) in powers of \( e \), \( D \) > 0 and check to see that for arbitrary values of the unknowns we have \( f = D \) for arbitrary values of the unknowns we have \( f = lim D \) for arbitrary values of the unknowns we have \( f = lim D \).

1204. The rank is an even number, the signature is zero.

1205. 
\[
\left[ \frac{n-1}{n} \right] + 1, \text{ where } [x] \text{ stands for the largest integer not exceeding } x. 
\]

1209. **Hint.** The proof is similar to that of the law of inertia.

1210. **Hint.** Make use of the preceding problem.

1211. The forms \( f_1 \) and \( f_2 \) are equivalent to each other but are not equivalent to the form \( f_3 \).

1212. The forms \( f_2 \) and \( f_3 \) are equivalent to each other but are not equivalent to the form \( f_1 \).

1213. In the complex domain \( n+1 \); in the real domain \( (n+1)(n+2) \) \( \frac{1}{2} \).

1214. The rank is an even number, the signature is zero.

1215. There are no such values of \( \lambda \). 

1216. There are no such values of \( \lambda \).

1217. **Hint.** Let \( y = f + P \), where \( f = c_1 x_1 + \ldots + c_n x_n \). Change the order of the unknowns so as to reach the case \( c_n \neq 0 \), perform the transformation 
\[
y_i = x_i \quad (i = 1, 2, \ldots, n-1), \\
y_n = \frac{x_n}{c_n}, 
\]
and prove that for the new forms \( D_4 = D_4 + c_n^2 D_{n-1} \), where \( D_{n-1} \) is a corner minor of order \( n-1 \) of the form \( f_1 \).

1218. **Hint.** Represent the form \( f \) as 
\[
l = a_{11} \left( x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \right)^2 + f_1 (x_1, \ldots, x_n) 
\]
and use the preceding problem to show that \( D_4 = a_{11} D_{f_1} + a_{11} D_{f_2} \). 

The inverse transformation is of the form: 
\[
x_1 = y_1 - y_2, x_2 = y_3 - y_4, \ldots; \\
x_{n-1} = y_{n-1} - y_n, x_n = y_1 + y_n. 
\]

**Hint.** Apply the transformation 
\[
x_1 = x_1 + x_2 + \ldots + x_n; \\
x_2 = x_2 + x_3 + \ldots + x_n; \\
\vdots \\
x_n = x_n. 
\]
Reducing both forms to the normal form, we find

\[ f = \sum_{i=1}^{n} y_i^2, \]

where \( y_i \) are linear forms in \( x_1, x_2, \ldots, x_n \). By virtue of the noted properties of a composition, we have \( (f, g) = \sum_{i=1}^{n} (y_i, z_i) \).

Consider one of the summands \( (y_i, z_i) \), where \( y_i = \sum_{k=1}^{n} a_{ik}x_k \cdot z_i = \sum_{k=1}^{n} b_{ik}x_k \cdot x_k = y_i \cdot z_i \cdot x_k \).

Then \( x_i = \sum_{k=1}^{n} b_{ik}x_k \cdot x_k \cdot (\sum_{k=1}^{n} a_{ik}x_k \cdot z_i)^2 \) is nonnegative for arbitrary real values of \( x_1, x_2, \ldots, x_n \). Hence, \( f = 0 \) and this proves \( (a) \). Now let \( f > 0 \) and \( g > 0 \). We reduce the form \( g \) to the normal form \( g = \sum_{i=1}^{n} y_i^2 \), where

\[ y_i = \sum_{k=1}^{n} a_{ik}x_k \quad (t = 1, \ldots, n) \quad \text{and} \quad |Q| = |q_{ij}| < 0. \]

Then \( (f, g) = \sum_{i=1}^{n} (y_i, z_i) \cdot x_k \cdot (\sum_{k=1}^{n} a_{ik}x_k \cdot z_i)^2 \) is nonnegative by virtue of \( f > 0 \). If for a certain \( t \) we take the value \( x_k \neq 0 \), then there is an \( \epsilon > 0 \) such that \( g | x_k | \cdot \epsilon > 0 \) (otherwise we would have \( Q = 0 \) between vertical bars). Hence, since \( f > 0 \), it is also true that \( (y_2, z_2) = \sum_{k=1}^{n} b_{ik}x_k \cdot x_k \cdot (\sum_{k=1}^{n} a_{ik}x_k \cdot z_i)^2 \neq 0 \) and \( (f, g) > 0 \).

1221. Hint. When proving \( (b) \), consider the forms \( f_k = \sum_{i=1}^{n} a_{ik}x_k \cdot x_k \cdot (\sum_{k=1}^{n} a_{ik}x_k \cdot z_i)^2 \) (\( k = 1, 2, \ldots, n \)).

1222. Hint. The necessity of the conditions (3) follows from the invariance of the corner minors under triangular transformations (see the preceding problem).

The equations (3) are proved in the same way. Sufficiency can be proved by induction on the number of unknowns \( n \).

1224. \( \eta_1 = -2y_2 + \frac{2}{3} y_3; \quad \eta_1 = y_1 + \frac{4}{3} \sqrt{3} y_2; \quad x_1 = y_1 + \frac{4}{3} \sqrt{3} y_2; \quad x_2 = \frac{1}{2} y_1 - \frac{4}{3} \sqrt{3} y_2 \).
1250. \[3y^3 + 6y^2 - 2y^2; \quad x_1 = \frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{2}}y_3; \quad x_2 = -\frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{2}}y_3; \quad x_3 = -\frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{2}}y_3,\]

1251. \[5y^3 - 2y^2 - 2y^2; \quad x_1 = \frac{1}{\sqrt{3}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{2}}y_3; \quad x_2 = \frac{1}{\sqrt{3}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{2}}y_3; \quad x_3 = \frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{2}}y_3,\]

1252. \[9y^3 + 18y^2 + 18y^2; \quad x_1 = \frac{1}{3}y_1 - \frac{2}{3}y_2 + \frac{2}{3}y_3; \quad x_2 = \frac{2}{3}y_1 + \frac{2}{3}y_2 + \frac{1}{3}y_3; \quad x_3 = \frac{1}{3}y_1 - \frac{2}{3}y_2 - \frac{1}{3}y_3,\]

1253. \[3y^3 - 6y^2; \quad x_1 = \frac{2}{3}y_1 + \frac{1}{6}V^2y_2 + \frac{1}{2}V^2y_3; \quad x_2 = \frac{1}{3}y_1 - \frac{2}{3}y_2 + \frac{1}{3}y_3,\]

1254. \[9y^2 + 9y^2 - 9y^2; \quad x_1 = \frac{2}{3}y_1 + \frac{1}{2}V^2y_2 + \frac{1}{6}V^2y_3; \quad x_2 = \frac{2}{3}y_1 - \frac{2}{3}y_2 + \frac{1}{6}V^2y_3,\]

1255. \[2y^2 + 4y^2 - 2y^2 - 4y^2; \quad x_1 = \frac{1}{2}(y_1 + y_2 + y_3 + y_4); \quad x_2 = \frac{1}{2}(-y_1 - y_2 + y_3 + y_4); \quad x_3 = \frac{1}{2}(-y_1 + y_2 - y_3 + y_4); \quad x_4 = \frac{1}{2}(y_1 - y_2 - y_3 - y_4),\]

1256. \[4y^3 + 8y^2 + 12y^2 - 4y^2; \quad x_1 = \frac{1}{2}(y_1 + y_2 + y_3 + y_4); \quad x_2 = \frac{1}{2}(y_1 - y_2 - y_3 + y_4); \quad x_3 = \frac{1}{2}(y_1 + y_2 - y_3 - y_4); \quad x_4 = \frac{1}{2}(y_1 - y_2 + y_3 + y_4).\]

1257. \[5y^2 - 3y^2 + 5y^2; \quad x_1 = \frac{1}{3}V_5(2y_1 + y_2); \quad x_2 = \frac{1}{5}V_5(y_1 + 2y_2); \quad x_3 = \frac{1}{5}V_5(-y_1 + 2y_2); \quad x_4 = \frac{1}{5}V_5(-y_1 - 2y_2).\]

1258. \[2y^2 - 4y^2; \quad x_1 = \frac{1}{2}V_2(y_1 + y_2); \quad x_2 = \frac{1}{2}V_2(y_1 - y_2); \quad x_3 = \frac{1}{2}V_2(y_1 + y_2); \quad x_4 = \frac{1}{2}V_2(y_1 - y_2).\]

1259. \[3y^2 + 6y^2 + 9y^2; \quad x_1 = y_1; \quad x_2 = \frac{1}{3}(y_2 + 2y_3 + 2y_4); \quad x_3 = \frac{1}{3}(2y_2 - 2y_3 + 2y_4).\]

1260. \[5y^2 + 5y^2 + 5y^2; \quad x_1 = \frac{1}{5}V_5(2y_1 + y_2); \quad x_2 = \frac{1}{5}V_5(-y_1 + 2y_2); \quad x_3 = \frac{1}{5}V_5(-y_1 - 2y_2); \quad x_4 = \frac{1}{5}V_5(y_1 + 2y_2); \quad x_5 = \frac{1}{5}V_5(-y_1 - 2y_2).\]

1261. \[4y^3 + 4y^3 + 4y^3 - 6y^2 - 6y^2; \quad x_1 = y_1; \quad x_2 = \frac{1}{5}V_5(y_1 + 2y_2); \quad x_3 = \frac{1}{5}V_5(-y_1 + 2y_2); \quad x_4 = \frac{1}{5}V_5(-y_1 - 2y_2); \quad x_5 = \frac{1}{5}V_5(y_1 - 2y_2).\]

1262. \[5y^2 - 5y^2 + 5y^2 - 5y^2 + 5y^2; \quad x_1 = \frac{1}{5}V_5(2y_1 + y_2); \quad x_2 = \frac{1}{5}V_5(-y_1 + 2y_2); \quad x_3 = \frac{1}{5}V_5(-y_1 - 2y_2); \quad x_4 = \frac{1}{5}V_5(y_1 + 2y_2); \quad x_5 = \frac{1}{5}V_5(-y_1 - 2y_2).\]

1263. \[\frac{1}{2}y^2 + \frac{1}{2}y^2 + \frac{1}{2}y^2 + \frac{1}{2}y^2; \quad y_1 = \frac{1}{\sqrt{2}}(y_1 + y_2 + \ldots + y_6).\]

1264. \[\frac{1}{2}y^2 - \frac{1}{2}y^2 - \frac{1}{2}y^2 - \frac{1}{2}y^2; \quad y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2 + \ldots + x_n).\]

1265. \[\text{Hint.} \quad \text{Show that under an orthogonal transformation of a quadratic form the characteristic polynomial of its matrix remains unchanged.}\]

1266. The forms $f$ and $h$ are orthogonally equivalent to each other but they are not orthogonally equivalent to the form $g$.

1267. The forms $g$ and $h$ are orthogonally equivalent to each other but they are not orthogonally equivalent to the form $f$.

1268. The forms $f$ and $h$ are orthogonally equivalent to each other but they are not orthogonally equivalent to the form $g$.

1269. \[Q = \begin{bmatrix} 2 & 3 & 3 & 3 & 3 \\ 2 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\]

1270. \[Q = \begin{bmatrix} 2 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\]

1271. \[\text{Hint.} \quad \text{Use the hint in problem 1074 to show that the eigenvalues of matrix } A - \lambda_0b^T \text{ are obtained by subtracting } \lambda_0 \text{ from the eigenvalues of matrix } A, \text{ and then apply problem 1242.}\]
1272. Hint. Use the preceding problem.

1274. A matrix with positive corner minors is orthogonal if and only if it is a unit matrix.

1275. Solution. The quadratic form with matrix $A^T A$ is positive definite (problem 1207); this means that a triangular transformation can reduce it to canonical form with positive coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal (all values of the roots are taken positive), and $B = D C$, then $A^T A = C D C' = B'B$, where the matrix $B$ satisfies the requirements of the problem.

Set $Q = A B^{-1}$. Then $Q^T Q = (A B^{-1})^T (A B^{-1}) = (B^{-1})^T B^{-1} B B^{-1} B = I$, that is, the matrix $Q$ is orthogonal and $A = Q B$, and II besides $A = Q B$, then the matrix $Q^{-1} Q = A^T A = B^T B$ is orthogonal and triangular with positive elements on the diagonal. This means that it is a unit matrix, whence $Q = Q_1$, and $B = B_1$.

1276. Solution. We prove assertion (a) for the representation $A = Q B$ of the required form. The matrix $A^T A$ is symmetric, and the quadratic form of the matrix is positive definite (problem 1207). Therefore there exists an orthogonal matrix $P$ such that $A^T A = P^T C P$, where $C$ is a diagonal matrix with positive elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal. Let $D$ be a diagonal matrix with the elements $\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \ldots, \frac{1}{\sqrt{\lambda_n}}$ on the diagonal (we take positive values of the root). Set $B = P^T D P = P^T D P$. From this it follows that $B$ is a symmetric matrix with positive eigenvalues. Hence the quadratic form with matrix $B$ is positive definite and the corner minors of it are positive. Furthermore, $A^T A = P^T C P = P^T D^T DP = P^T D^T D P = B^T B$. Set $Q = A B^{-1}$. Then $Q^T Q = A^T A = B^T B = B^T B = E$, which means that the matrix $Q$ is orthogonal.

Suppose we have two representations of the required type: $A = Q_1 B_1 = Q_2 B_2$. Then $A^T A = B_1^T B_1 = B_2^T B_2$. Let us denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues, respectively, of $A$, $A^T A$ and $D^T D$, which are all positive and $\lambda_i = \lambda_j$ for $i = 1, 2, \ldots, n$ (problem 1277).

Hence, $\mu_1 = \nu_1 (i = 1, 2, \ldots, n)$. Let $C$ and $D$ be diagonal matrices with elements $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ and $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ on the diagonal. There are orthogonal matrices $U$ and $V$ such that $B_1 = U V C$, $B_2 = W V C$, $W^T = V^T D V$, $U^T = U^T C$, $C^T = V^T C$, $V^T U^T = U^T C$, $C^T U^T = U^T C$, $U^T = U^T C$, $V^T U^T = U^T C$.

The matrix $W (w_{ij}) = U V$ commutes with $L$. Let us now show that it commutes with $D$. If $L = \lambda$, then, by comparing the element of the matrix $C W = W C$ in the $i$th row and $j$th column, we will find that $w_{ij} = 0$. Thus, if we represent the matrix $L$ in the form of a block-diagonal matrix with diagonal blocks $U_1, C_2, \ldots, C_n$, so that in each block the diagonal elements are the same, and are distinct in different blocks, then the matrix $W$ is a block-diagonal matrix with diagonal blocks $W_1, C_2, \ldots, C_n$. By the construction of the matrix $D$, there will be a block-diagonal matrix with the diagonal blocks $D_1, D_2, \ldots, D_k$ of the same size and with identical diagonal elements.

The proof is shorter if one uses functions of matrices. If $A$ is the desired representation, then $A^T A = B^2$, hence $A^T A$, and the eigenvalues of $B$ are positive. Hence, $B$ is of the form $\lambda A$, and the eigenvalues of the matrix $A^T A$ are positive, and $A$ is orthogonal, namely, there is a polynomial of the symmetric matrix $A^T A$, which is a symmetric matrix (problems 1148, 1151). Putting $\lambda = A^T A$, we see (as we did above) that $Q$ is orthogonal.

The representation $A = B^2$ is obtained in similar fashion with the aid of the matrix $A^T A$. The assertion (b) is proved in the same way as (a) with the positive definite forms replaced by Hermitian positive definite forms. Assertion (c) follows from the uniqueness of the representations given in (a) and (b). It can also be demonstrated by reducing matrix $B$ to diagonal form in the case (1) and matrix $A$ in the case (2) via an orthogonal (unitary) matrix (see problem 1596).

Chapter IV. Vector Spaces and Their Linear Transformations

1277. (1, 2, 3). 1278. (1, 1, 1). 1279. (0, 2, 1, 2).

1280. $x_1 = -2x_2 - 7x_3 - 41x_4; x_2 = 9x_1 + 20x_3 + 9x_4; x_3 = 4x_1 + 13x_2 + 8x_3.$

1281. $x_1 = 2x_2 - x_3 - x_4; x_2 = -3x_1 + x_3 - 2x_4; x_3 = x_1 + 3x_3.$

1282. (a) $a_0, a_1, a_2, \ldots, a_n$. (b) $f(x), f'(x), f''(x), \ldots, f^{(n)}(x).$

1283. 

\[
\begin{pmatrix}
1 - \alpha & \alpha^2 & \alpha^3 & \cdots & (1-n)\alpha^n \\
0 & 1 - 2\alpha & 3\alpha^2 & \cdots & (n-3)\alpha^{n-2} \\
0 & 0 & 1 - 3\alpha & \cdots & (n-4)\alpha^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 - \alpha^n
\end{pmatrix}
\]

In this matrix, the $k$-th column involves the number $(-\alpha)^k C_{n-k}^k - (-\alpha)^{k-1} C_{n-k+1}^k$. The proof is similar to the proof of (1274). (a) Two rows are interchanged; (b) two columns are interchanged; (c) there will be a symmetric reflection of the matrix about its center.

1284. (a) No. 1286. (b) No.

1287. Yes, if the given straight line passes through the origin, if it does not.

1288. Yes. 1290. No. 1291. Yes.

1292. No. 1293. Yes.

1294. The entire space; vectors lying in any plane that passes through the coordinate origin; vectors on any straight line that passes through the origin, and the origin itself, that is, the single zero vector.

1295. No. 1296. No.

1297. The basis is formed, for example, by the vectors $(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0, 0), (0, 0, 1, \ldots, 0, 0), \ldots, (0, 0, 0, \ldots, 1).$ The dimension is equal to $n$. 
The basis is formed by the following vectors: if $k$ is the number of a basis vector, then its coordinate numbered $2k - 1$ is equal to 1, and the other coordinates are equal to zero, $k = 1, 2, \ldots$ 

\[
\left\lfloor \frac{n+1}{2} \right\rfloor
\]

where $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$.

The dimension is equal to $\left\lfloor \frac{n+1}{2} \right\rfloor$.

The basis is formed by vectors that are designated as basis vectors in the answer to the preceding problem, and another vector is adjoined whose coordinates with even numbers are equal to unity, those with odd numbers are equal to zero. The dimension is equal to $\frac{n+1}{2} + 1$.

The basis is formed, for example, by the matrices $E_{ij}$, where $E_{ij}$ is a matrix whose element in the $i$th row and the $j$th column is equal to unity; all other elements are zero. The dimension is equal to $n^2$.

The basis is formed, for example, by the polynomials $x^i, x^j, \ldots, x^n$. The dimension is equal to $n+1$.

The basis is formed, for example, by the matrices $F_{ij}$, where $F_{ij}$ is a matrix whose elements $F_{ij} = f_{ij} = 1$ and all other elements are zero. The dimension is equal to $\frac{n(n+1)}{2}$.

The dimension is equal to $\frac{n+1}{2}$.

The dimension is equal to $n^2$.

The dimension is equal to $n+1$.

The dimension is equal to $n-1$.

The dimension is equal to $n-1$.

The dimension is equal to $n+1$.

The dimension is equal to $n+1$.

The dimension is equal to $\frac{n(n+1)}{2}$.

The dimension is equal to $\frac{n(n-1)}{2}$.

The dimension is equal to $\frac{n(n+1)}{2}$.

The dimension is equal to $\frac{n-1}{n}$.

The projection on $L_1$ parallel to $L_2$ has all coordinates equal to $\frac{1}{n}$.

\[
A_1 = 3 \quad \frac{1}{2} \quad 1 \quad \frac{1}{2} \quad 1 \quad 3 \\
A_2 = -\frac{1}{2} \quad 0 \quad 1 \quad -\frac{1}{2} \quad 0
\]

Consider the parametric equations of the straight lines $x = a_0 + a_1 t, x = b_0 + b_1 t$, where $a_0, a_1, b_0, b_1$ are given vectors.

The vectors $a_0 - b_0, a_1, b_1$ must be linearly independent.

The desired conditions are that the vectors $a_0$ and $b_1$ are linearly independent, and the vector $a_0 - b_0$ can be expressed linearly in terms of $a_0$ and $b_1$. If $a_0 - b_0 = t_1 a_1 + t_2 b_1$, then the point of intersection is given by the vector $x = t_1 a_1 + t_2 b_1$.

The necessary and sufficient conditions are that the quadruplet of vectors $a_0 - c, b_0 - c, a_1, b_1$ is linearly dependent, and each of the two triplets $a_0 - c, a_1, b_1$ and $b_0 - c, a_1, b_1$ is linearly independent. The desired straight line has the equation $x = c + dt$, where $d = \lambda_1 (a_0 - c) + \lambda_2 a_1 = \lambda_3 (b_0 - c) + \lambda_4 b_1$ and the coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are nonzero.
The points of intersection of this straight line and the given straight lines are of the form
\[ a_5 = \frac{\lambda_5}{\lambda_5} a_1 + \frac{1}{\lambda_5} c \quad \text{and} \quad b_5 = \frac{\lambda_5}{\lambda_5} b_1 + \frac{1}{\lambda_5} c. \]

1340. If \( c = d t \), where \( d = (0, 1, -1, -1); M_1 (2, 2, -3, -4); M_2 (-4, 5, 5, 5), \)
1341. \( x = c + dt \), where \( d = (1, 1, 1, 3); M_3 (2, 3, 2, 1), \)
1342. \( M_4 (1, 2, 2, -2). \)

1344. If two planes of three-dimensional space have a common point, they have a common straight line. If a plane and a three-dimensional linear manifold of four-dimensional space have a common point, then they have a common straight line. If two three-dimensional linear manifolds of four-dimensional space have a common point, they have a plane in common.

1345. For convenience in classifying different cases we introduce two matrices: \( A \) is a matrix in which the columns contain the coordinates of the vectors \( a_1, a_2, b_1, b_2, \) and \( B \) is a matrix obtained from \( A \) by adjoining a column of coordinates of the vector \( a_0 - b_0. \) Let the rank of \( A \) be \( r_1 \) and the rank of \( B \) be \( r_2. \) One of six cases is possible:
1. \( r_1 = 4, r_2 = 5. \) The planes do not lie in a single four-dimensional manifold (the planes are absolutely skew).
2. \( r_1 = r_2 = 4. \) The planes have a common point and, hence, in one four-dimensional manifold, but do not lie in one three-dimensional manifold (the planes intersect absolutely).
3. \( r_1 = 3, r_2 = 4. \) The planes do not have any common points. They lie in one four-dimensional manifold, but do not lie in one three-dimensional manifold (the planes are skew parallel).
4. \( r_1 = r_2 = 3. \) The planes lie in a three-dimensional space and intersect along a straight line.
5. \( r_1 = r_2 = 3. \) The planes do not have any points in common, but they lie in one three-dimensional space (the planes are parallel).

1346. Hint. Carry out the proof by induction on the number \( k. \)

1347. A tetrahedron with vertices at the points
\[ (1, 1, -1, -1), (1, -1, 1, -1), (-1, 1, 1, -1), (-1, -1, 1, -1). \]

Hint. Note, in determining the coordinates of the vertices, that the vertices of the desired section must be the points of intersection of the cutting subspace with the edges of the cube, and that these coordinates along each edge of the cube are equal to \( -1, 1, -1, 0, \) while the fourth one varies from \( -1 \) to \( 1. \)

1349. A tetrahedron with vertices at the points
\[ \left( \frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}; \right), \left( -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}; \right), \left( -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}; \right), \left( -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, -\frac{1}{4}; \right). \]

Hint. Find the projections of the vertices.

1350. Hint. Take the given end of the diagonal for the origin, take the edges issuing from it for the coordinate axes and shew that the parallel linear manifolds under consideration are determined by the equations
\[ x_1 + x_2 + \ldots + x_6 = k \ (k = 0, 1, 2, \ldots, n) \]
while the point of intersection of the diagonal with the \( k \)th of these manifolds have all coordinates equal to one and the same number \( \frac{k}{n}. \)

1352. The bilinear form \( g \) must be symmetric, that is, \( a_{i j} = a_{j i} \) \( (i, j = 1, 2, \ldots, n) \) and the corresponding quadratic form \( f = 0 \)
\[ \sum_{i=1}^{n} a_{i j} x_i x_j \text{ must be positive definite; } (e_i, e_j) = a_{i j} \ (i, j = 1, 2, \ldots, n). \]

1354. They may be completed by adjoining the vectors \( (2, 2, 1, 0), (5, 1, 2, -1). \)

1355. They may be completed by adjoining the vectors \( (1, -2, 1, 0), (25, 4, -17, -6). \)

1358. Hint. Obtain the proof by induction on \( k. \)
The distance between the centres of the faces, it will also be the distance between the faces. Squaring the expression for \( z \) and then taking the root, we get
\[
|z| = \sqrt{\frac{2}{n+1} \left( a^2 - b^2 \right)^2 + \frac{2}{n-1} \left( b^2 - c^2 \right)^2}.
\]

1379. Hint. Find \( m \) from the condition \((x - m, e) = 0\). When proving uniqueness, form the scalar product of both sides of the equation \( a_1 e + x = a_2 e + z \) into \( e \).

1380. Hint. Consider the scalar square of the vector \( x = x_1 - \sum_{i=1}^{n-1} a_i e_i \), where \( a_1 = (x, e_1) \), and apply the properties (c) and (d) of the preceding problem.

1381. First method: consider the scalar square \((x + ty, x + ty)\) as a nonnegative quadratic trinomial in \( t \). Second method: for \( y \neq 0 \) represent \( x \) as \( x = ay + z \), where \((y, z) = 0\), show that \((x, x) \geq a^2 (y, y)\), equality occurring if and only if \( x = ay \), and determine
\[
(x, y)^2 - a^2 (y, y) (y, y) \leq (x, x) (y, y).
\]
Third method: apply the inequality of problem 505 to the coordinates of the vectors \( x \) and \( y \) in an orthonormal basis.

1382. Hint. First method: consider the scalar square \((x + ty, x + ty)\), where \( t = s(x, y) \) as a nonnegative quadratic trinomial in \( s \) (\( s \) real). Second method: for \( y \neq 0 \) put \( \alpha = ay + z \), where \( \alpha \) is a complex number and \((y, z) = 0\); show that \((x, x) \geq a (y, y)\), equality occurring if and only if \( x = ay \), and find out that
\[
(x, y) (y, x) = a (x, y) (y, y) \leq (x, x) (y, y).
\]
Third method: apply the inequality of problem 505 to the coordinates of the vectors \( x \) and \( y \) in an orthonormal basis.

1384. \( \left( \int_a^b f(x) g(x) \, dx \right)^2 \leq \int_a^b |f(x)|^2 \, dx \cdot \int_a^b |g(x)|^2 \, dx. \)

1385. \( AB = BC = AC = 6; \angle A = \angle B = \angle C = 60^\circ. \)

1386. \( AB = 5; BC = 10; AC = 3 \sqrt{3}; \angle A = 60^\circ; \angle B = 90^\circ; \angle C = 30^\circ. \)

1389. Hint. If the edges are given by the vectors \( a_1, a_2, \ldots, a_n \), then consider the expression \(|a_1 + a_2 + \ldots + a_n|^2 \). Second method: for \( y \neq 0 \) put \( \alpha = ay + z \), where \( \alpha \) is a complex number, \((y, z) = 0\); show that equality occurring if and only if \( x = ay \), and find out that
\[
(x, y) (y, x) = a (x, y) (y, y) \leq (x, x) (y, y).
\]
Third method: apply the inequality of problem 505 to the coordinates of the vectors \( x \) and \( y \) in an orthonormal basis.

1394. \( a \sqrt{n} \lim_{n \to \infty} \gamma_n = \infty. \)

1395. \( \theta_n = \arccos \frac{1}{\sqrt{n}} ; \lim \theta_n = \frac{\pi}{2} ; \theta_n = 60^\circ. \)

1396. \( H = \frac{a}{2} \sqrt{n} \). \( R < a \) for \( n = 4, 3; R = a \) for \( n = 4 \), and
\( R > a \) for \( n > 4. \)
Show that the origin of a diagonal may be joined to any other vertex by a chain of edges and make use of the preceding problem.

Hint. Take advantage of problem 1379.

1400. Solution. \( \cos(x, y) = \frac{(x, y)}{|x| |y|} = \frac{|x|}{|x| |y|} \cdot \frac{|y|}{|x| |y|} \cdot \cos(x, y) \leq \frac{|y|}{|x|} = -\cos(x, y). \)

Equality is possible if and only if \( \cos(y, y') = 1 \), that is, according to problem 1380, when \( y' = \alpha \cdot y \) or \( \alpha > 0 \).

1401. \( \arccos \frac{2}{\pi} \). 1402. \( \arccos \frac{2}{3} \). 1403. \( \arccos \frac{2}{4} \).

1405. \( \arccos \frac{2}{3} \). Hint. Let \( \alpha_i \) be a vector from \( A_i \) to \( A_i \) \((i = 1, 2, 3, 4)\). Consider the two vectors \( \alpha_i + \alpha_2 \) and \( \alpha_3 + \alpha_4 \); show that the square of the cosine of the angle between them is equal to \( \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \), and find the maximum of the function \( f(x) = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \), provided that \( \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = 1 \).

1406. \( \arccos \frac{2}{3} \). Hint. Seek the minimum of the angles of the vectors of the second plane with their orthogonal projections on the first plane.

1407. \( \arccos \frac{2}{3} \). Hint. Show that each of the systems \( f_1, \ldots, f_k \) and \( g_1, \ldots, g_k \) is a basis of the subspace \( L_n \) spanned by the vectors \( e_1, \ldots, e_n \) that \( \langle f_i, g_j \rangle = 0 \) for \( i, j \). Finally, that all coefficients \( c_1, c_2, \ldots, c_n \) are zero in the equation \( \sum (c_i) = c_1f_1 + \cdots + c_kf_k \).

1408. \( \arccos \frac{2}{3} \). Hint. Set \( (z^2 - 1)^k = u_k(z) \) and verify that \( u_k(1) = 0 \) for \( j < k \); then integrate \( \int_1^1 u_k^2(z) z^j \, dz \) by parts several times until the factor of the form \( z^j \) vanishes under the integral sign and show that this integral is equal to zero for \( j = 0, 1, \ldots, k - 1 \). From this derive the required equality: \( \int_1^1 P_j(z) P_k(z) \, dz = 0 \) for \( j = k \).

1409. \( P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2} (3x^2 - 1), P_3(x) = \frac{1}{2} (5x^3 - 3x), \)

\( P_4(x) = \frac{1}{5} (35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{2} \sum_{j=0}^{k} (-1)^{k-j} \binom{2}{j} \binom{2j}{k-j} x^{k-j} = \sum_{j=0}^{k} \binom{2}{j} \binom{2j}{k-j} x^{k-j} \). In these expressions, drop all terms with negative exponents of \( z \).

1410. \( \int \sqrt{x^2 + 1} \). Solution. \( \int \sqrt{x^2 + 1} \, dx = \frac{1}{2} \left[ \sqrt{x^2 + 1} \right] + C \). Integrating by parts, we find \( \frac{1}{2} \left[ (x^2 + 1)^k \right] + C \) and compute the scalar square \( (P_k, P_k) \). Integrating by parts, we find

\[ \int_1^1 u_k(x) u_k^{(n)}(x) \, dx = \int_1^1 u_k^{(k-1)}(x) u_k^{(n+k-1)}(x) \, dx \]

\[ \cdots = (-1)^k \int_1^1 u_k(x) u_k^{(2k)}(x) \, dx = 2k \int_1^1 x^k (1 + x^k) \, dx. \]

Again integrating by parts, we find

\[ \int_1^1 (1 - x)^k (1 + x)^k \, dx = \frac{k}{k+1} \int_1^1 (1 - x)^{k-1} (1 + x)^{k+1} \, dx \]

\[ \cdots = \frac{k!}{(k+1)(k+2)\cdots(2k)} \int_1^1 (1 + x)^{2k} \, dx = \frac{2k!}{(2k+1)k}. \]

whence

\( (P_k, P_k) = \frac{1}{2^{2k}} \frac{(2k)!}{2k+1} \).

1411. \( P_k(1) = 1. \) Hint. Apply to the expression \( P_k(x) = \frac{1}{2^{2k}} \frac{(2k)!}{2k+1} \) the Leibniz rule for differentiating a product \( \frac{d^k}{dx^k} [x^k + 1]^k \) of \( P_k(x) \) the Leibniz rule for differentiating a product \( \frac{dx^k}{dx^k} [x^k + 1]^k \) the Leibniz rule for differentiating a product

\[ \frac{d^k}{dx^k} [x^k + 1]^k \]

1412. \( P_k(x) = C_k f_k(x) \), where \( C_k = \frac{2k!}{2^{2k}} \frac{(2k)!}{2k+1} \).

Hint. Make use of the problems 1407, 1408, 1409.

1414. \( \int \frac{dx}{(x+1)^k} \). Solution. \( \int \frac{dx}{(x+1)^k} = \frac{1}{k-1} \frac{1}{(x+1)^{k-1}} \). Integrate by parts, the bar stands for replacing elements by complex conjugate
1419. Hint. First method: show that the Gram determinant \( g(a_1, \ldots, a_k) \) is equal to the square of the modulus of the determinant formed up to the coordinates of the vectors \( a_1, \ldots, a_k \) in any orthonormal basis of a \( k \)-dimensional subspace containing those vectors.

Second method: show that the nonnegative quadratic form \( (a_1 z_1 + \cdots + a_k z_k) \) in \( x_1, \ldots, x_k \) is positive definite if and only if the vectors \( a_1, \ldots, a_k \) are linearly independent.

Third method: using the invariance of the Gram determinant when orthonormalizing vectors (problem 1415), show that if the vectors \( b_1, \ldots, b_k \) are obtained from \( a_1, \ldots, a_k \) by the process of orthonormalization, then \( g(a_1, \ldots, a_k) = \|b_1\|^2 \cdots \|b_k\|^2 \), and apply problem 1413.

1421. \( \frac{1}{2n(n+1)} \). Hint. First method: noting that the desired distance is equal to the length of the orthogonal component of the vector \(-x^i\) relative to the subspace of polynomials of degree not exceeding \( i-1 \), apply the preceding problem and also problem 418.

Second method (it dispenses with problem 418): the desired distance yields the minimum of the integral \( \int |f(x)|^2 \, dx \), where \( f(x) \) is an \( n \)-th-degree polynomial with leading coefficient unity. This enables one, by changing the limits of integration, to reduce the problem to the appropriate extremal property of the Legendre polynomial (problem 1414).

1422. Hint. Apply problems 1413 and 1415.

1423. \[ |D^2| \leq \left( \sum_{i=1}^{n} a_{ij}^2 \right) \left( \sum_{i=1}^{n} b_{ij}^2 \right), \] equality occurring if and only if either \( \sum_{i=1}^{n} a_{ij}b_{jk} = 0 \) \( (i \neq j; i, j = 1, 2, \ldots, n) \) or the determinant \( D \) contains a zero row.

1424. Hint. Introduce into the vector space \( K_n \) the scalar product \( (x, y) = \sum_{i=1}^{n} a_{ij}y_j \), where \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) are the coordinates of \( x \) and \( y \) respectively in some basis \( e_1, \ldots, e_n \) of \( K_n \); show that \( D_{ij} = g(e_i, \ldots, e_n) \) and apply problem 1422.

1425. Hint. Use problem 1419 (d).

1426. Hint. Use the properties of Hermitian forms similar to the properties of the real forms indicated in problem 1310 [see, for example, F. R. Gantmakher's "The Theory of Matrices", Chelsea, New York, 1959].

1427. Hint. Make use of the reasoning in the first and third methods given in the answer to problem 1415.


1429. Solution. Let the process of orthogonalization carry the vectors \( a_1, \ldots, a_k, b_1, \ldots, b_k \) into the vectors \( c_1, \ldots, c_k \) and the vectors \( b_1, \ldots, b_k \) into the vectors \( d_1, \ldots, d_k \) relative to the subspace \( L \) spanned by \( a_1, \ldots, a_k, b_1, \ldots, b_{k-1} \) coincides with \( L \).

Indeed, \( b_i = y_i + z_i \), where \( y_i \) can be linearly expressed in terms of \( b_1, \ldots, b_{k-1} \) and \( c_i \) is orthogonal to these vectors. \( a_i = y_i + x \), where \( y_i \in L_i \) and \( x \) is orthogonal to \( L_i \), but then \( b_i = (y_i + y_i) + x \), where \( y_i + y_i \in L_i \) and \( z \) is orthogonal to \( L_i \). Hence, by problem 1413, \( z = b_1 + \cdots + b_{k-1} \) and, hence, is orthogonal to \( c_1, \ldots, c_k \); \( b_1, \ldots, b_k \) and by problem 1415 we have

\[ g(a_1, \ldots, a_k, b_1, \ldots, b_k) = |c_1|^2 \cdots |c_k|^2 \]

which proves inequality (1). Under the condition (2), \( |d_j| \leq |c_j| \) \( (j = 1, 2, \ldots, n) \) and inequality (1) becomes an equality. If \( a_1, \ldots, a_k \) or \( b_1, \ldots, b_k \) are linearly dependent, then the right member of inequality (1) vanishes, but since the left member is nonnegative, we again obtain an equality.

Conversely, let the inequality (1) become an equality. Then by what has already been demonstrated \( |c_j| = \cdots = |c_k| = |d_j| = \cdots = |d_k| = 0 \) \( (j = 1, 2, \ldots, n) \), whence there either exists an \( i \leq k \) such that \( |e_i| = 0 \), that is, \( a_1, \ldots, a_k \) are linearly dependent, or there exists an \( i \leq k \) such that \( |d_i| = |e_i| = 0 \) \( (i = 1, 2, \ldots, n) \), by which it follows that all \( a_i \) and, hence, all \( b_i \) (as linear combinations of them) are orthogonal to \( a_1, \ldots, a_k \), which means condition (2) is fulfilled.

1430. Hint. Use the preceding problem.


1433. Solution. (a) Let the numbers of system (1) be the distances of all possible pairs of vertices \( M_0, M_1, \ldots, M_n \) of an \( n \)-dimensional simplex, and suppose

\[ a_{ij} = M_iM_j \quad (i, j = 0, 1, 2, \ldots, n; i > j). \]

Denote by \( c_1 \) the vector from \( M_0 \) to \( M_i \) \( (i = 1, 2, \ldots, n) \). We then have

\[ a_{ie} = (c_1, c_i) \quad (i = 1, 2, \ldots, n) \]

\[ a_{ji} = (c_i - c_j, c_i - c_j) \quad (i, j = 1, 2, \ldots, n; i > j) \]

From these equalities we find the scalar products of the vectors

\[ (c_1, c_i) = a_{i0} \quad (i = 1, 2, \ldots, n) \]

\[ (c_i, c_j) = \frac{a_{ii} + a_{ji} - a_{ij}}{2} \quad (i, j = 1, 2, \ldots, (i > j)) \].
Using these relations, we can write down the Gram matrix of the vectors $e_1, \ldots, e_n$:
\[
\begin{pmatrix}
\frac{a_{10} + a_{11} - a_{12}}{2} & \frac{a_{20} + a_{22} - a_{21}}{2} & \cdots & \frac{a_{n0} + a_{nn} - a_{n1}}{2} \\
\frac{a_{20} + a_{21} - a_{22}}{2} & a_{10} & \cdots & \frac{a_{n0} + a_{n1} - a_{n2}}{2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a_{n0} + a_{n1} - a_{n2}}{2} & \frac{a_{n2} + a_{n3} - a_{n4}}{2} & \cdots & a_{n0}
\end{pmatrix}
\]  
(5)

Since the points $M_0, M_1, \ldots, M_n$ do not lie in an $(n-1)$-dimensional manifold, the vectors $e_1, e_2, \ldots, e_n$ are linearly independent. Denote by $D_k$ the corner minor of order $k$ of the matrix (5) and apply problem 1419 to get
\[
D_k > 0 \quad (k = 1, 2, \ldots, n).
\]  
(6)

Thus, conditions (6) are the necessary conditions for the numbers of system (1) to be the distances of the vertices of an $n$-dimensional simplex. We now show that these conditions are sufficient. When conditions (6) are fulfilled, the matrix (5) is the Gram matrix of a linearly independent set of vectors $e_1, \ldots, e_n$ [see problem 1352 or 1310 (c)]. Thus equalities (4) hold, whence follow equalities (3). Taking $M_0$ for the origin of coordinates and $M_i$ for the terminus of the vector $e_i$ ($i = 1, 2, \ldots, n$), we see that equalities (2) hold, which is what we set out to prove.

In the case (b), a necessary and sufficient condition is that all principal minors (not only corner minors) of matrix (5) be nonnegative. The necessity is demonstrated in the same way as in the case (a) with the sole difference that the vectors $e_1, \ldots, e_n$ may be linearly dependent. Sufficiency is demonstrated with the aid of problem 1425 or 1210 (d).

1434. \[
\begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}
\]

1435. \[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]  if $e_1$ goes into $e_2$ and \[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  if $e_2$ goes into $e_1$.

1436. \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

1437. \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

1438. \[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

1439. The first $k$ elements of the main diagonal of the transformation matrix are equal to unity, all other elements being zero.
The th and jth rows and the ith and jth columns in the matrix are interchanged.

\[
\begin{pmatrix}
1 & 0 & 2 & 1 \\
2 & 3 & 5 & 1 \\
3 & -1 & 0 & 2 \\
1 & 1 & 2 & 3
\end{pmatrix}
\quad \begin{pmatrix}
2 & -1 & 0 & 1 \\
1 & 4 & -3 & 8 \\
4 & 1 & 6 & 4 \\
1 & 3 & 4 & 7
\end{pmatrix}
\]

1452. (a) \[
\begin{pmatrix}
0 & 2 & 0 \\
1 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
2 & 3 & 5 \\
3 & 4 & 2 \\
1 & 1 & 2
\end{pmatrix}
\]

1453. \(A = 2 \) and \(A = 3\).

1454. \(A = 4\).

1457. \[
\begin{pmatrix}
-20 & -44 \\
-23 & -25
\end{pmatrix}
\]

1458. \[
\begin{pmatrix}
-10 & 93 \\
34 & 23
\end{pmatrix}
\]

1461. \(G = 2\).

1464. Hint. Use the preceding problem and also the definition of the function of a matrix (Problem 1441).

1465. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2 \\
s_3
\end{pmatrix}
\]
where \(s_1, s_2, s_3 \neq 0\).

1466. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1467. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1468. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1469. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1470. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1471. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1472. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1473. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1474. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1475. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1476. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1477. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1478. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1479. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1480. The matrix cannot be reduced to diagonal form.

1481. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1482. The matrix cannot be reduced to diagonal form.

1483. \(A = 2\) and \(A = 3\). The eigenvectors are of the form 
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
where \(s_1, s_2 \neq 0\).

1484. For example,

\[
T = \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

where along the secondary diagonal the elements above the main diagonal are \(-1\) and below are \(+1\); when \(n\) is odd, both \(+1\) and \(-1\) can reside at the intersection of the diagonals. The diagonal matrix \(H\) has, above the main diagonal, \(n/2\) elements (for even \(n\)) and \((n+1)/2\) elements (for odd \(n\)) equal to \(+1\), the remaining elements being equal to \(-1\).

1485. Hint. Consider a linear transformation \(q\) of \(n\)-dimensional space having the matrix \(A\) in a certain basis, and the basis consisting of the eigenvectors of the transformation, and apply Problem 1477. 1486. Any two elements \(a_{ij}\) and \(a_{k\ell}\) equidistant from the ends of the secondary diagonal either must both be nonzero or must both vanish.

1487. The sole eigenvalue is \(A = 0\); the eigenvectors are polynomials of zero degree.

1488. Solution. First prove the equality

\[\dim L = \dim \eta L + \dim \eta L_0.\]

where \(L\) is the intersection of \(L\) and the kernel \(\eta L\) of the transformation \(\eta\). To do this, first complete the basis \(\eta_1, \eta_2, \ldots, \eta_n\) of the subspace \(\eta L_0\) (using the vectors \(b_1, b_2, \ldots, b_n\)) to form a basis of \(\eta L\) (for \(\eta L_0 = 0\) the vectors \(b_i\) are absent, for \(\eta L_0 = L\) the vectors \(b_i\) are absent). The vectors \(\eta b_1, \eta b_2, \ldots, \eta b_n\) form a basis of \(\eta L\). Indeed, if \(\eta x = \eta y\), then \(x = \eta x = \eta y\), where \(x \in L\). If \(x = \sum \eta_1 b_1 + \sum \eta_2 b_2\), then \(x = \eta x = \sum \eta_1 b_1 = \sum \eta_2 b_2\), since \(\eta b_1 = 0\).
The vectors \( \varphi \), \( \varphi_2, \ldots, \varphi_n \) are linearly independent since

\[
\sum_{i=1}^{n} \beta_i \varphi_i = 0 \implies \sum_{i=1}^{n} \beta_i \varphi_i = 0 \quad \text{where} \quad \sum_{i=1}^{n} \beta_i = 0
\]

Thus, \( \dim L = l + \kappa = \dim \varphi L + \dim L_0 \).

By (2) from (4), by virtue of \( \varphi_0 \),

\[
\dim L = \dim \varphi L + \dim L_0 \leq \dim \varphi L + \text{nullity } \varphi;
\]

\[
\dim L = \text{nullity } \varphi \leq \dim \varphi L
\]

Furthermore, \( \dim \varphi L = \dim L - \dim L_0 \leq \dim L \).

Set \( \varphi^{-1}L = L' \). Since \( L \subseteq \varphi^{-1}L \), then \( \varphi^{-1}L \subseteq \varphi^{-1}L \). Applying (1) with \( L' \) substituted for \( L \), we get

\[
\dim L' \leq \dim \varphi L' + \text{nullity } \varphi
\]

Since \( \varphi L' \subseteq L \), it follows that \( \dim \varphi L' \leq \dim L \) and by (2) \( \dim L' \leq \dim L + \text{nullity } \varphi \), which thus proves the second of the inequalities of (b).

Now let us show that \( \varphi L' = L \cap \varphi R_0 \).

Since \( \varphi L' \subseteq L \cap \varphi R_0 \), and \( \varphi L' \subseteq \varphi R_0 \), then \( \varphi L' \subseteq L \cap \varphi R_0 \). If \( x \in L \cap \varphi R_0 \), then \( x = x \varphi^{-1}L \), where \( x \in \varphi^{-1}L \), that is, \( x \in \varphi L' \), whence \( L \cap \varphi R_0 \subseteq \varphi L' \).

Since \( \dim (L + \varphi R_0) \leq n \), we get, using the relationship between the dimension of a sum and an intersection of subspaces (problem 1316),

\[
\dim \varphi L' = \dim (L \cap \varphi R_0) = \dim L + \dim \varphi R_0 - \dim (L + \varphi R_0)
\]

\[
\geq \dim L + \dim \varphi R_0 - n = \dim L - \text{nullity } \varphi.
\]

From this, via (2), we obtain

\[
\dim L' = \dim \varphi L' + \text{nullity } \varphi \geq (\dim L - \text{nullity } \varphi) + \text{nullity } \varphi = \dim L
\]

and this completes the proof of the first of the inequalities of (b).

1492. Hint. Consider the transformations \( \varphi \) and \( \Psi \) of the space \( R^k \), with matrices \( A \) and \( B \) and apply the preceding problem to the subspace \( L = \varphi R^k \).

1494. The sole eigenvalue is \( \lambda = 1 \). The eigenvectors have the form \( \alpha + \alpha_2 \alpha_2 + \cdots + \alpha_n \alpha_n \), where \( \alpha_1, \alpha_2 \) are not simultaneously zero.

1495. Hint. Consider the matrix of the transformation \( \varphi \) in a basis whose first vectors are linearly independent eigenvectors of \( \varphi \) associated with \( \lambda \). An alternative approach is by using problem 1504.

1501. The zero subspace and the subspace \( L_0 \), which consist of all polynomials of degree \( \leq \kappa \), consist of the invariant subspaces: the zero subspace and any subspace spanned by an arbitrary subsystem of vectors of the basis \( \alpha_0, \alpha_1, \ldots, \alpha_n \), their total number is \( 2^n \).

Hint. Use problems 1497 and 1502 to show that any nonzero invariant subspace \( L \) has a basis which is a subset of the vectors \( \alpha_1, \alpha_2, \ldots, \alpha_n \).

1496. The sole eigenvalue is \( \lambda = 1 \). The eigenvectors have the form \( \alpha + \alpha_2 \alpha_2 + \cdots + \alpha_n \alpha_n \), where \( \alpha_1, \alpha_2 \) are not simultaneously zero.

1507. If the minimal polynomial \( \varphi (A) = \lambda^n - c_1 \lambda^{n-1} - c_2 \lambda^{n-2} - \cdots - c_n \), then the transformation matrix \( \varphi \) is of the form

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

1518. The Jordan submatrix

\[
\begin{pmatrix}
\alpha & 1 & 0 & \cdots & 0 \\
0 & \alpha & 1 & \cdots & 0 \\
0 & 0 & \alpha & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \alpha
\end{pmatrix}
\]
1525. \( g(\lambda) = (\lambda - 2)(\lambda - 3) \), \( B_2 = L_1 \oplus L_2 \), where \( L_1 \) has, for example, the basis \( f_1 = e_1 + e_2 \), \( f_2 = e_1 - e_3 \) and \( L_2 \) the basis \( e_2 - 5e_2 - 6e_3 \).

1526. \( g(\lambda) = [\lambda - (a, a)] \lambda \), \( B_2 = L_1 \oplus L_2 \), where \( L_1 \) is spanned by the vector \( a \) and \( L_2 \) is formed by all vectors orthogonal to \( a \).

1527. If \( \lambda \) is an eigenvalue of \( g \), then the Jordan form consists of a single block of order \( n \) with \( \lambda \) on the diagonal.

1529. Solution. (A). Set \( f_i(\lambda) = \frac{i}{\lambda} \) \( (i=1, 2, \ldots, s) \). The polynomials \( f_i(\lambda) \) are relatively prime. Hence, there exist polynomials \( h_i(\lambda) (i=1, 2, \ldots, s) \) such that \( 1 = \sum_{i=1}^{s} f_i(\lambda) h_i(\lambda) \), whence for any vector \( x \),

\[
x = \sum_{i=1}^{s} x_i h_i(\lambda),
\]

where \( x_i = f_i(\psi) h_i(\psi) \) since \( (\psi - \lambda) f_i(\psi) x_i = f_i(\psi) h_i(\psi) x_i = 0 \) by the Hamilton-Cayley theorem, by virtue of which \( f_i(\psi) = 0 \).

It suffices to prove the uniqueness of the expansion (1) for \( x = 0 \). Applying the transformation \( f_i(\psi) \) to both sides of the equality \( \sum_{i=0}^{s} x_i = 0 \), we get \( f_i(\psi) x_i = 0 \) since \( f_i(\psi) x_i = 0 \) for \( \psi \neq i \). Furthermore, \( f_i(\lambda) \) and \( (\lambda - \lambda_i)^{h_i} \) are relatively prime. Hence, there exist polynomials \( p(\lambda) \) and \( q(\lambda) \) such that

\[
1 = p(\lambda) f_i(\lambda) + q(\lambda) (\lambda - \lambda_i)^{h_i},
\]

whence

\[
x_i = p(\psi) f_i(\psi) x_i + q(\psi) (\psi - \lambda_i)^{h_i} x_i = 0.
\]

This proves that the space \( R_n \) is a direct sum of the subspaces \( P_i (i=1, 2, \ldots, s) \) and the construction of the desired basis has been reduced to the case (B). In the case of a minimal polynomial, the reasoning is similar (see problem 1523).

(B) It suffices to prove that the indicated construction is possible at every step (that is, the vectors that serve to complete the basis of \( R_{n-1} \) are linearly independent) and that the vectors of all constructed sets form a basis for \( R_n \). To each set in the transformation matrix \( \Psi \) there obviously corresponds a Jordan submatrix with zero, and in the transformation matrix \( \Phi = \lambda_0 \Psi + \Phi \) there corresponds a Jordan submatrix with \( \lambda_0 \) on the diagonal.

The possibility of construction at each step is proved inductively for \( h = k, k - 1, \ldots, 1 \). When \( h = k \), the vectors of any basis of \( R_{k-1} \) together with the vectors of the height \( k \) that start the sets of the \( k \)-step form, by construction, a basis of \( R_k \). Let us suppose that the sets with initial vectors of height \( \geq k + 1 \) have already been constructed, and all vectors of height \( h+1 \) of the constructed sets \( x_i, \ldots, x_p \), together with the vectors \( y_1, \ldots, y_k \) of any basis of \( R_k \) form a basis of \( R_{k+1} \). We will show that the vectors of height \( h \) \( y_{k+1}, \ldots, y_p \) of the constructed sets together with any basis \( z_1, \ldots, z_r \) for \( R_{k+1} \) are linearly independent.

Applying the transformation \( \psi^h \) to both sides of this equality,

\[
\sum_{i=1}^{p} c_i x_i + \sum_{j=1}^{r} d_j z_j = 0
\]

we get \( \sum_{i=1}^{p} c_i x_i = 0 \) and can be linearly expressed in terms of its basis \( y_1, \ldots, y_p \). From the linear independence of the vectors \( x_i, \ldots, x_p \) (as a basis of \( R_{k+1} \) it follows that \( c_i = 0 \) \( (i=1, 2, \ldots, r) \). For this reason, from the equality (2) it follows that \( d_j = 0 \) \( (j=1, 2, \ldots, r) \), this proves that the vectors of height \( h \) of the earlier constructed sets together with the vectors of any basis of \( R_{k+1} \) may be completed to a basis of \( R_k \). Taking the supplementary vectors (if they exist) for the initial vectors of new sets, we find that the assumption made above for \( R_{k+1} \) is now fulfilled for \( R_k \), and the construction may be continued.

Suppose the indicated construction has been carried out for \( h = k, k - 1, \ldots, 1 \) (actually, the basis of the entire space may result before we arrive at \( h = 1 \)). Since \( R_k \) does not have a basis, it follows from what has been proved that vectors of height \( i \) of the constructed sets form a basis for \( R_i \); hence these vectors, together with vectors of height 2 of the constructed sets, form a basis of \( R_3 \), and so forth. Finally, vectors of height \( k = 1 \) of the constructed sets, together with vectors of height \( k \) of these sets, form a basis of \( R_k \). In other words, the vectors of all constructed sets form a basis of the entire space.

(C) Let \( C = A_h - \lambda_h E \). Since the matrices \( B^h \) and \( C^h \) are similar, the rank of \( C^h \) is equal to \( r_h \) \( (h=0, 1, \ldots, k, k+1) \). To each Jordan submatrix of matrix \( A_h \) with \( \lambda_h \) on the diagonal there corresponds in matrix \( C \) a block of the same order with zero on the diagonal. If \( D \) is such a block of order \( p \), then the rank of block \( D^h \) for \( h = 0, 1, 2, \ldots, p \) is equal to \( p - h \), and for \( h = p+1, \ldots, k+1 \) it is equal to zero. To the block with \( \lambda_h = \lambda_0 \) of matrix \( A_h \) there corresponds, in matrix \( C \), a block with the number \( \lambda_h - \lambda_0 \) on the diagonal. The rank of any power of it is equal to its order. The rank of the matrix \( C^h \) is equal to the sum of the ranks of its blocks. Therefore, when passing from matrix \( C^{h-1} \) to matrix \( C^h \) the rank is lowered by exactly the number of blocks of matrix \( C \) with zero on the diagonal, the blocks having orders \( \geq h \). Hence

\[
\sum_{i=1}^{k} x_i = r_i - r_{i-1} \quad (i=1, 2, \ldots, k)
\]

Subtracting from this a similar equality with \( h + 1 \) substituted for \( h \) (for \( h < k \)), we obtain the relations (C) for \( h = 1, 2, \ldots, k-1 \). Since there are no blocks of orders exceeding \( k \) with \( \lambda_0 \) on the diagonal in matrix \( A_h \), it follows that \( r_h = r_{h+1} \), and for \( h = k \) the relation (3) is fulfilled.
which means that $rot(\text{in rotation axis})$ holds for $\mathbf{A}$ as well.

1531. $I_2 = (1, 3, 1, 1), \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

1532. $I_3 = (1, 3, 1, 1), \quad A_3 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

1533. $I_4 = (3, 1, 1, 1), \quad A_4 = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

1534. $I_5 = (1, 1, 1, 1, 1), \quad A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

1535. $I_6 = (0, 1, 1, 1, 0), \quad A_6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

1536. For an even $n$:

$$I_1 = e_1, \quad I_2 = e_2, \quad I_3 = e_3, \quad \ldots, \quad I_n = e_n.$$  

$$I_{n+1} = e_2, \quad \ldots, \quad I_{2n+1} = e_n.$$  

The matrix $A_1$ consists of two Jordan submatrices of orders $n$ with zero on the main diagonal. For an odd $n$:

$$I_1 = e_1, \quad I_2 = e_2, \quad I_3 = e_3, \quad \ldots, \quad I_{n+1} = e_n, \quad \ldots, \quad I_{n+1} = e_n.$$  

$$I_{n+2} = e_2, \quad \ldots, \quad I_{2n+1} = e_n.$$  

The matrix $A_1$ consists of two Jordan submatrices of orders $n+1$ with zero on the main diagonal.

1537. $q$ is a reflection of the space $R_n$ in some subspace $L_1$ parallel to some supplementary subspace $L_2$. In other words, $R_n$ is a direct sum of $L_1$ and $L_2$ with $q(x) = x$ if $x \in L_1$ and $q(x) = -x$ if $x \in L_2$.
1562. For example, the transformation \( q \) that carries the vector 
\[ \mathbf{v} = (x_1, x_2, \ldots, x_k), \]
specified by coordinates in an orthonormal basis, into the vector 
\[ \mathbf{p} = (y_1, y_2, \ldots, y_k), \]
preserves scalar squares but is not linear.

Hint. To prove the linearity of \( q \), show that
\[
(q(ax + by) - axq - bvy, \quad q(ax + by) - axq - bvy) = 0
\]

1563. (a) \( UA^{-1} = A'U \), (b) \( U\bar{A}^{-1} = A'U \).

1566. Hint. Show that there exists a vector \( \mathbf{x} \) for which
\[ (x_1', x_1) = 0 \] (that may be zero) for which \( (x_1', x_1) = 0 \) (that may be zero) for which
and that by putting
\[ U'' = U - \sum_{i=1}^{h-1} x_i y_i \]
we get systems of vectors \( x_1, \ldots, x_{h-1}, \ldots, x_k, y_1, \ldots, y_{h-1}, y_h \) with equal Gram matrices; then apply the method of mathematical induction.


1568. Solution. (a) If \( q \mathbf{x} = \lambda \mathbf{x} \), then
\[ (q \mathbf{x}, q \mathbf{x}) = \lambda (x, x) = \lambda x^2 \]
whence \( \lambda = 1 \) and \( |x| = 1 \) if \( \lambda \neq 1 \).

(b) Let \( q \mathbf{x} = \lambda_1 \mathbf{x} \), \( q \mathbf{y} = \lambda_2 \mathbf{y} \), \( \lambda_1 \neq \lambda_2 \). Then
\[ (x_1, x_2) = (y_1, y_2) = \lambda_1 \lambda_2 (x_1, x_2), \]
whence, multiplying by \( \lambda_2 \) and taking into account that \( \lambda \lambda_2 = 1 \), we find \( \lambda_1 (x_1, x_2) = \lambda_1 (x_1, x_2) \) and, hence, \( (x_1, x_2) = 0 \).

(c) Let \( X \) and \( Y \) be the columns of the coordinates \( x \) and \( y \). Passing to coordinates in the equation \( \langle x + yi \rangle = (x + y) i \), we get \( AX = AY \). Hence, \( A = (X - Y, Y - X) i \), whence, multiplying by \( i \) and taking into account the real and imaginary parts, we obtain \( AX = \alpha X + \beta Y; \quad AY = -\beta X + \alpha Y \), and that proves the equations (1). Multiplying termwise the first of equations (1) by itself and applying the relation \( \alpha^2 + \beta^2 = 1 \), we get
\[ (\mathbf{x}, \mathbf{x}) = \alpha x^2 + \beta y^2 \]
whence, multiplying together equations (1), we get
\[ (\mathbf{x}, \mathbf{y}) = (\alpha x + \beta y, \alpha x + \beta y) = \alpha \beta (|x|^2 - |y|^2) + \alpha \beta (\alpha x + \beta y), \]
Finally, cancelling out \( \beta \), we obtain the following system of equations for \( \beta \): \( \beta (|x|^2 - |y|^2) + 2\alpha (x, y) = 0 \), \( \alpha (|x|^2 - |y|^2) - 2\beta (x, y) = 0 \).

Since the determinant of the system is nonzero, it follows that
\[ |x|^2 - |y|^2 = 0 \] and \( (x, y) = 0 \).

(d) If \( \phi \) has a real eigenvalue, then there is a one-dimensional invariant subspace, otherwise we pass to a unitary space. Namely, we take the orthonormal basis \( e_1, \ldots, e_k \) in the unitary space \( H_k \). The vectors of \( H_k \) that have real coordinates in that basis form a Euclidean space \( H_k \) embedded in \( H_k \). The transformation \( \phi \) has, in the basis \( e_1, \ldots, e_k \), \( \phi \) is an orthogonal matrix \( \mathbf{A} \). In the given basis, this matrix determines the unitary transformation \( \phi' \) which coincides with \( \phi \) on \( H_k \). The transformation \( \phi' \) has an eigenvalue \( \alpha + \beta i \), where \( \beta \neq 0 \). If \( x + yi \) is the corresponding eigenvector and \( x, y \) are vectors with real coordinates, then equalities (1) hold. Thus, the subspace of the space \( H_k \), which subspace is spanned by the vectors \( x \) and \( y \), is invariant under \( \phi \).

1570. (a) For any unitary matrix \( A \), there exists a unitary matrix \( Q \) such that the matrix \( Q^* AQ \) is a diagonal matrix with elements on the diagonal equal to unity in modulus.

(b) The space \( H_k \) is a direct sum of pairwise orthogonal one-dimensional and two-dimensional subspaces that are invariant under \( \phi \). The transformation \( \phi \) leaves the vectors of one-dimensional subspaces invariant or inverts them; on a two-dimensional subspace it produces a rotation through the angle \( \gamma \). For any orthogonal matrix \( A \), there exists an orthogonal matrix \( Q \) such that the matrix \( Q^* AQ \) is in the canonical form indicated in the problem.

Hint. Make use of problems 1567 and 1569 and apply the method of mathematical induction.

1571. \[ f_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \]
\[ f_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0, \quad -1 \end{pmatrix}, \]
\[ f_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1, \quad -2, \quad 1 \end{pmatrix}, \]
1572. \[ f_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, \quad 1, \quad 0 \end{pmatrix}, \]
\[ f_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, \quad 0, \quad 1 \end{pmatrix}, \]
\[ f_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, \quad -1, \quad 0 \end{pmatrix}, \]
\[ f_4 = \begin{pmatrix} 0, \quad 0, \quad 0 \end{pmatrix}. \]
1573. \( f_1 = \frac{1}{\sqrt{42 + 28V^2}} \begin{pmatrix} -2 - V^2 \end{pmatrix}, \)
\( f_2 = \frac{1}{\sqrt{V^2}} \begin{pmatrix} 0 \end{pmatrix}, \)
\( f_3 = \frac{1}{\sqrt{84}} \begin{pmatrix} 0, \sqrt{2 - 28} \end{pmatrix}. \)

\( B = \begin{bmatrix} 1 & 0 & 0 \\ -2 + 7V^2 & 12 & 12 \\ 0 & 42 + 28V^2 & -2 + 7V^2 \end{bmatrix}. \)

1574. \( B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & V^2 \\ 0 & -\frac{V^2}{2} & \frac{1}{2} \end{pmatrix}, \)

\( Q = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \)

1575. \( B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \frac{V^2}{2} \\ 0 & -\frac{V^2}{2} & \frac{1}{2} \end{pmatrix}, \)

\( Q = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \frac{V^2}{2} \\ 0 & -\frac{V^2}{2} & 0 \end{pmatrix}. \)

1576. \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}. \)

1577. \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

1578. \[
\begin{pmatrix}
\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}
\end{pmatrix}
\]

1584. \( \phi \) is either an identical transformation or a symmetry about subspace \( L \) of dimension \( k = 0, 1, 2, \ldots, n - 1; \) that is, \( \phi x = x \) for any \( x \) in \( L \) and \( \phi x = -x \) for any \( x \) in its orthogonal complement \( L^s. \)

1585. \[
\begin{vmatrix}
\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{vmatrix}
\]

1586. \[
\begin{vmatrix}
\frac{1}{V^2} & \frac{1}{V^2} & 0 \\
\frac{1}{V^2} & \frac{1}{V^2} & 0 \\
\frac{1}{V^2} & \frac{1}{V^2} & 0
\end{vmatrix}
\]
1587. \[ f_1 = \frac{1}{\sqrt{2}} (1, -i, 0), \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \]
\[ f_2 = \frac{1}{\sqrt{2}} (1, i, 0), \quad f_3 = (0, 0, 1); \]
\[ B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1-i & 1+i \\ \sqrt{3} & \sqrt{6} \\ \sqrt{2} & \sqrt{6} \end{pmatrix}. \]

1589. \[ B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 2-i \\ -1 & 2+i \end{pmatrix}. \]

1590. Hint. Let \( E_{ij} \) be a matrix in which the \( i \)th row and the \( j \)th column have unity, with zero elsewhere. Consider the matrices of all transformations in an orthonormal basis:
\[ E_{11}, E_{21}, \ldots, E_{n1}, E_{12}, E_{22}, \ldots, E_{n2}, \ldots, E_{nn}. \]

1591. (a) \( UA = A'U \), (b) \( U^T = A'U \).

1592. Two real quadratic forms can be brought to canonical form by one and the same orthogonal transformation if and only if their matrices commute. Two second order surfaces have parallel principal axes if and only if the matrices of the coefficients of the squared terms in their equations are permutable. Hint. Prove that the subspace \( L \) of all eigenvectors of the transformation \( \varphi \) that belong to one and the same eigenvalue \( \lambda \) is invariant under a second transformation \( \psi \).

1593. If
\[ A_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} b_{i1} \\ b_{i2} \\ \vdots \\ b_{in} \end{pmatrix}, \]
\((i = 1, 2, \ldots, n)\) are columns that are orthonormal eigenvectors of \( P \) and \( Q \) respectively, then \( a \) matrices \( X_{ij} = A_i B_j^T \) \((i = 1, 2, \ldots, n)\) \( J = 1, 2, \ldots, n) \), where in the \( k \)th row and the \( l \)th column of matrix \( X_{ij} \) we have the product \( a_{ij} b_j \). Form an orthonormal basis of eigenvectors of the transformations \( \varphi \) and \( \psi \); note that any such basis is obtained in the indicated manner from certain orthonormal bases of eigenvectors of the matrices \( P \) and \( Q \).

1594. If, for example, a linear transformation \( \varphi \) of a plane in an orthonormal basis is specified by the matrix \( \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \) and the vector \( x = (1, 1) \) is given by coordinates in the same basis, then \( x \langle 2, 2 \rangle = -4. \)

1595. Hint. Make use of the problem 1276 (c). It is easier to consider the transformation matrices \( \varphi, \psi, \chi \) in an orthonormal basis of eigenvectors \( e \) of the transformation \( \chi \).

1596. Hint. The existence is proved as in problem 1276. Uniqueness is more easily proved by using the preceding problem.

1597. The eigenvalues of the transformation with matrix \( \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \) are equal to 3, -4, which means they are not both positive.

1598. \[ \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \sqrt{3} \\ \frac{1}{2} & \frac{3}{2} & \sqrt{3} \\ \frac{5}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & \sqrt{2} \\ \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} & \sqrt{3} \\ \frac{3}{2} & \frac{3}{2} & \sqrt{3} \\ \frac{5}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & -\sqrt{2} \\ \frac{1}{2} \end{pmatrix}. \]

1599. \[ \begin{pmatrix} 14 & 2 & -4 \\ 3 & 3 & 3 \\ 2 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 3 \\ 3 & 3 & -3 \\ 3 & 3 & 3 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 14 \\ 3 & 3 & 3 \\ -3 & 3 & 3 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 2 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}. \]

1600. Hint. Find a nonsingular transformation \( \chi \) such that \( \chi^T \psi \psi = \psi \) and show that the transformation \( \chi^T \psi \psi \psi \) is self-conjugate.

1601. Hint. Let \( \psi_1 \) and \( \psi_2 \) be self-conjugate transformations with nonnegative eigenvalues such that \( \psi_1^T = \psi_1 \) and \( \psi_2^T = \psi_2 \). If \( \psi \) is non-singular, then, putting \( \psi = \varphi \psi \), show that \( \chi^T \psi \psi \psi \).

1602. Hint. Consider an orthonormal basis in which the matrix of \( \varphi \) is diagonal and make a transition to the new basis.

1603. Hint. Show the distributivity of the transition operation from \( A \) to \( B \). Consider the linear transformations \( \varphi, \psi, \chi \) specified in some orthonormal basis of a unitary space \( \mathcal{H} \) by the matrices \( A, B, C \). Split the transformations \( \varphi, \psi, \chi \) into sums of nonnegative self-conjugate transformations of rank 1 with matrices \( A_1, A_2, A_3, A_4 \). Use problem 1606 to show that the transformation with matrix \( (A_1, A_2, A_3, A_4) \) in the same basis is nonnegative. Finally, use problem 1604 to show that the transformation \( \chi \) is nonnegative.

1604. Hint. The proof is similar to that of the appropriate property of a self-conjugate transformation.

1605. Hint. Consider an orthonormal basis in which the matrix of \( \varphi \) is diagonal and make a transition to the new basis.

1606. Show the distributivity of the transition operation from \( A \) to \( B \). Consider the linear transformations \( \varphi, \psi, \chi \) specified in some orthonormal basis of a unitary space \( \mathcal{H} \) by the matrices \( A, B, C \). Split the transformations \( \varphi, \psi, \chi \) into sums of nonnegative self-conjugate transformations of rank 1 with matrices \( A_1, A_2, A_3, A_4 \). Use problem 1606 to show that the transformation with matrix \( (A_1, A_2, A_3, A_4) \) in the same basis is nonnegative. Finally, use problem 1604 to show that the transformation \( \chi \) is nonnegative.

1607. Hint. The proof is similar to that of the appropriate property of a self-conjugate transformation.

1608. The properties (a) and (b) are proved in the same way as the corresponding properties of a self-conjugate transformation.

1609. Proof of property (c): Let \( X \) and \( Y \) be the columns of coordinates of \( x \) and \( y \) respectively. Passing to coordinates in the equality \( \varphi(x + y) = \varphi(x) + \varphi(y) \), we get: \( AX + AY = -BY + BX \). whence,
equating real and imaginary parts, we find: $AX = -BY$, $AY = \beta X$, which is proof of equality (1). Since the matrix $A$ is real, it follows that the vectors $z$ and the numbers $(p, q_1, q_2)$ are real for the real vector $z$. Therefore, $(pq, x) = (z, -qz) = (z, qz) = (qz, x)$ and, hence, the numbers $(p, q_1, q_2)$ are real for the real vector $z$. Multiplying the second of equalities, we find $(qz, x) = - (z, qz)$ and, hence, $x = - (z, qz)$. Multiplying the first by $y$ and the second by $x$ and then adding, we find $x = - (z, qz)$ and $y = - (z, qz)$, whence $|x| = |y|$. The numbers $(p, q_1, q_2)$ are real for the real vector $z$. Multiplying the second of equalities by $y$, we find $(pq, x) = (z, qz) = (p, q_z) = -(z, qz)$ and, hence, $x = - (z, qz)$. Multiplying the first by $y$ and the second by $x$ and then adding, we find $x = - (z, qz)$ and $y = - (z, qz)$, whence $|x| = |y|$. The numbers $(p, q_1, q_2)$ are real for the real vector $z$. Proof of property (d): if the transformation $\Phi$ has 0 as an eigenvalue, then there is a one-dimensional invariant subspace, otherwise we pass to a unitary space. Namely, in the unitary space $\mathbb{R}^n$, we take an orthogonal basis $e_1, \ldots, e_n$. The vectors in $\mathbb{R}^n$ that have real coordinates in that basis form a Euclidean space $\mathbb{R}^n$, embedded in $\mathbb{R}^n$. In the basis $e_1, \ldots, e_n$, the transformation $\Phi$ has a real skew-symmetric matrix $A$. In the given basis, this matrix defines the skew-symmetric transformation $\Phi'$ of the unitary space $\mathbb{R}^n$, which coincides with $\Phi$ on $\mathbb{R}^n$. The transformation $\Phi'$ carries vectors of one-dimensional subspaces into zero, and on the two-dimensional subspace that corresponds to the block

\[
\begin{pmatrix}
0 & \beta \\
-\beta & 0
\end{pmatrix}
\]

it causes a rotation through the angle $\beta$ with multiplication by the number $-\beta$. For any real skew-symmetric matrix $A$ there exists a real orthogonal matrix $Q$ such that the matrix $B = Q^{\dagger}AQ$ is diagonal with pure imaginary elements on the diagonal, some of which may be zero.

1616. If $\Phi$ is a skew-symmetric transformation, then the matrix $E(\Phi)$ is unitary (respectively orthogonal). If $\Phi$ is a skew-symmetric matrix, then $E(\Phi)$ is a skew-symmetric transformation. Besides, $\Phi$ can be expressed in terms of $\Phi$ with the aid of an equality similar to (1).

1617. Solution. By problem 1539, the transformation $\Phi$ is self-conjugate. If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\Phi$, then they are real, and, by problem 1616, the eigenvalues of $E(\Phi)$ are $\lambda_1, \ldots, \lambda_n$, that is, $E$ is positive definite. Let us now show that to distinct self-conjugate transformations $\Phi$ and $\Phi'$ there correspond distinct transformations $E(\Phi)$ and $E(\Phi')$. Let $\Phi' = E(\Phi)$; $\Phi$ has an orthonormal basis of eigenvectors $a_1, \ldots, a_n$, where $\varrho_1 = \lambda_1 a_1$ (1 = 1, 2, $\ldots$, n). Let $a'$ be any eigenvector of the transformation $\Phi'$ with the eigenvalue $\lambda'; a' = \sum x_i a_i$. Then, by problem 1466, $E(\Phi') a' = E(\Phi) a' = \sum x_i \varrho_i a_i$. On the other hand, $E(\Phi') a' = \sum x_i \varrho_i a_i$, $\sum x_i \varrho_i a_i$; since $E(\Phi') a' = E(\Phi) a'$, it follows that $x_i = 0$ for all those $i$ for which $\lambda_i \neq \lambda_j$ and, $\lambda_j \neq \lambda_i$, if $x_i \neq 0$. Since $\lambda_j$ and $\lambda_i$ are real, it follows from $x_i = 0$ that $\lambda_j \neq \lambda_i$. Therefore, $\varrho_i = \sum x_i \varrho_i a_i = \sum x_i \lambda_i a_i$; since $E(\Phi') a' = E(\Phi) a'$, it follows that $\varrho_i = \lambda_i a_i$. Let $\Phi$ be any positive definite transformation. Then an orthonormal basis in which the matrix $\Phi$ is diagonal with positive elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal. Suppose $\lambda_0$ = $\lambda_0$ ($\lambda_0 = 1, 2, \ldots, n$, where $\lambda_0$ is the real value of the orthonormal basis $a_1, a_2, \ldots, a_n$ on the diagonal. The transformation $\Phi$ is self-conjugate and $\Phi = E(\Phi')$. 1625. Hint. Apply problem 1507. 1626. If $\Phi$ is the rank of $\Phi$, then the number of such transformations is equal to $k^2$. 1627. Hint. Prove the equality

\[
(\varrho - \lambda x, \varrho - \lambda x) = (\varrho^2 x - \lambda x, \varrho^2 x - \lambda x).
\]
1628. Hint. Use the preceding problem.
1629. Hint. Use problem 1627.
1630. Hint. Make use of problems 1627 and 1629.
1631. Hint. Apply problem 1629 several times.
1632. Hint. To prove necessity, use the two preceding problems.

To prove sufficiency, show that the transformation \( \phi \) of unitary space, which transformation has the normal property, has an orthonormal basis of eigenvectors. Reduce the case of Euclidean space to that of unitary space.

1633. Hint. Use the preceding problem.

**Supplement**


1637. Hint. First method: show that \( |a| = 1 \) for any \( a \) of the given group \( G \) of order \( n \). For \( n > 1 \), take, in \( G \), an element \( b = \cos \psi + i \sin \psi \) with least positive argument \( \psi \) and show that

\[
G = \{1, b, b^2, \ldots, b^{n-1}\}.
\]

Second method: using the Lagrange theorem, show that \( a^n = 1 \) for any \( a \) in \( G \).

1638. (a) One group—a cyclic group of the third order—with elements \( e, a, b \) and the array

\[
\begin{array}{c|ccc}
  & e & a & b \\
  e & e & a & b \\
a & a & b & e \\
b & b & e & a \\
\end{array}
\]

In the representation as substitutions, we can set \( e \) for the unit element, \( a = (1 2 3), b = (1 3 2) \).

(b) Two groups: (1) a cyclic group of the fourth order with elements \( e, a, b, c \) and the array

\[
\begin{array}{c|cccc}
  & e & a & b & c \\
  e & e & a & b & c \\
a & a & b & c & e \\
b & b & c & e & a \\
c & c & e & a & b \\
\end{array}
\]

In the representation as substitutions, we can set \( e \) for the unit element, \( a = (1 2 3 4), b = (1 3 2 4) \).

(c) Two groups: (1) a cyclic group of sixth order with elements \( e, a, b, c, d, f \) and the array

\[
\begin{array}{c|cccccc}
  & e & a & b & c & d & f \\
  e & e & a & b & c & d & f \\
a & a & b & c & d & f & e \\
b & b & c & d & f & e & a \\
c & c & d & f & e & a & b \\
d & d & f & e & a & b & c \\
f & f & e & a & b & c & d \\
\end{array}
\]

In the representation as substitutions, we can set \( e \) for the unit element, \( a = (1 2 3 4 5 6), b = (1 3 5 2 4 6), c = (1 4 2 5 3 6), d = (1 5 3 2 4 6), f = (1 6 5 4 3 2) \).

(2) a symmetric group of the third degree with elements \( e, a, b, c, d, f \) and the array

\[
\begin{array}{c|cccccc}
  & e & a & b & c & d & f \\
  e & e & a & b & c & d & f \\
a & a & b & c & d & f & e \\
b & b & c & d & f & e & a \\
c & c & d & f & e & a & b \\
d & d & f & e & a & b & c \\
f & f & e & a & b & c & d \\
\end{array}
\]

In the representation as substitutions, we can set \( e \) for the unit element, \( a = (1 2 3), b = (1 3 2), c = (1 2), d = (2 3), f = (1 3) \).

Hint. Show that if in a group \( G \) of order \( n \) there is a set \( H \) of \( k \) elements, \( k < n \), which is itself a group under the operation of multiplication specified in \( G \), then by multiplying all elements in \( H \) by the element \( x \) not in \( H \), we obtain \( k \) new elements of \( G \). For that reason \( k < \frac{n}{2} \). For \( H \) we can take the set of elements \( e, a, a^2, \ldots, a^{k-1} \), where \( a^k = e \). For example, in the case (2) of (c), that is, for a non-cyclic group \( G \) of sixth order, it must be true that \( k \leq 3 \). If it were true that \( a^2 = e \) for any \( a \) in \( G \), then the four elements \( e, a, a^2 \)
would form a group, but that is impossible. Hence there is an element $a$ for
which $a 
eq e$ but $a^2 = e$. Multiplying the elements $a, a, a^2$ by the new element $e$, we obtain all six elements of the group $G$ in the form $e, a, a^2 = b, c, d = a^3 = f$. It must be shown that $e^2 = f^2 = e^3 = f$. For example, if it were true that $e^2 = f^2$, then by multiplying on the left first by $e$ and then by $a^2$, we would have $e^2a^2 = f = e$, whence $f^2 = e^2 = f$, which is impossible.

1639. The tetrahedron group $G$ is of order 12, the cube and octahedron of order 24, the dodecahedron and icosahedron of order 60. Hint. Consider rotations that carry the given vertex $A$ into some vertex $B$ (not necessarily different from $A$) and show that the order of the group is equal to $nk$, where $n$ is the number of vertices and $k$ is the number of edges emanating from one vertex.

1643. Hint. With every element $x$ of the given group $G$ associate a mapping $a 	o ax$ for every element $a$ of $G$.

1648. Hint. Consider $(ab)^p$ and $(ab)^q$, where $p$ is the order of $ab$, $(b)$. Consider $(ab)^p$, where $p$ is the order of $ab$, and show that $a^p = b^p = e$.

Example 2. Concerning the elements $a 
eq e, b = a^{-1}$, condition (1) is fulfilled but condition (2) is not. The assertion (b) is not fulfilled since the orders of $a$ and $b$ are equal to each other, but are not equal to the order of $ab = a$.

Example 3. In the cyclic group $G$ of order 8, the elements $a, a^2, a^3$ are of order 4, but $a^4 = e$ is of order 2, and $a^5 = e$ is of order 4.

Example 4. Let $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$. Then $G$ is a group with four elements, but it is not a cyclic group.

Example 5. The group $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$, is a cyclic group of order 4.

Example 6. The group $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$, is a cyclic group of order 4.

Example 7. The group $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$, is a cyclic group of order 4.

1653. The symmetric group $S_n$ has $n!$ elements, and hence it is a group. The group $G$ contains all cyclic groups of order $n!$.

1654. (a) $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$.

(b) $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$.

(c) $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$.

(d) $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$.

(e) $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$.

(f) $G = \{a, a^2, a^3, a^4\}$, where $a^4 = e$.

1655. In $G$ we choose any elements: first $a \neq e$, then $b \neq a, a, e$, and then $c \neq e, a, b, ab$. Then the remaining elements of group $G$ are $ab, ac, bc, abc, G$ is an Abelian group (problem 1652). The group $G$ has the following 16 subgroups: $\{e\}, \{a, b\}, \{a, b, e\}, \{a, ab\}, \{a, ba\}, \{e, ab, b\}, \{e, b, ba\}, \{e, a, ab\}, \{e, a, ba\}, \{e, a, b, ab\}, \{e, a, b, ba\}, \{e, a, ab, ba\}, \{e, a, b, ab, ba\}$.

1657. Written additively, all subgroups are of the form $G_0 = \{a\}, G_1 = \{pa\}, G_2 = \{p^2a\}, \ldots, G_{n-1} = \{p^{n-1}a\}, G_n = \{p^n\} = \{e\}$.

They form a decreasing chain of subgroups of the orders, respectively, $p^n, p^{n-1}, p^{n-2}, \ldots, p, 1$.


(c) Hint. Verify the product of two transpositions can be expressed in the following manner in terms of triple cycles: $(1)(2)(3)(4) = (1)(2)(3)(4)$.

(d) Solution. Let $G$ be a subgroup of the alternating group $A_n$ generated by the set of the indicated triple cycles, and let $i, j, k$ be distinct numbers exceeding two (for $n = 3$ the assertion is obvious and for $n = 4$ the proof given below is reduced). Together with the cycle $(123)$ the group $G$ contains the inverse element $(231)$, and then $G$ contains

$\{e, (123), (132), (231), (321), (123), (132), (231)\} = \{e\}$.

For $n = 4$ the group $G$ contains all triple cycles. For $n > 4$ it contains $(123), (132), (142)(13)(2)(14)(3)(2)(14)(3)(2)$, coincides with $A_n$.

1660. Hint. Let $K$ be the set of all elements of a group $G$ which do not belong to $H$, and let $a$ be any element of $K$. Show that by multiplying $a$ by all elements of $H$, we obtain all elements of $K$. From this an automatic consequence is that by multiplying $a$ by all elements taken from $K$, we obtain all elements of $H$. In particular an example is the group $G$ with elements $a, b, c$, where $a^2 = b, c, ab$ (see answer to problem 1638). It has three cyclic subgroups of the second order: $\{a, b\}$ and \{c\}. $\{a\}$

1661. (a) Hint. To every rotation of the tetrahedron $ABCD$ there corresponds a substitution of its vertices. To a product of two rotations there corresponds a product of the respective substitutions. Two distinct rotations $s$ and $t$ correspond to different substitutions with the identical substitution that mantains all vertices fixed.
From the answer to problem 1639, the tetrahedron group is isomorphic to a subgroup of twelfth order of the symmetric group $S_4$. Now, one can either verify that all substitutions corresponding to rotations of the tetrahedron are even or one may take advantage of problem 1674.1.

(b) Solution. The centres of the faces of the dodecahedron are vertices of a cube. Therefore the groups of the cube and the dodecahedron are isomorphic. To each rotation of the cube there corresponds a substitution of its four diagonals. A product of rotations is associated with a product of the respective substitutions. Let us consider all rotations of the cube. These are: the identical rotation, eight rotations about the diagonals through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, six rotations about the axes passing through the midpoints of the opposite edges through the angle $\pi$, and nine rotations about the axes, passing through the centres of the opposite faces, through the angles $\frac{\pi}{4}$, $\frac{2\pi}{4}$, and $\frac{3\pi}{4}$. The number of these rotations comes to $1 + 8 + 6 + 9 = 24$. According to the answer to problem 1639, these exhaust all rotations of the cube. A direct verification shows that only under the identical rotation do all four diagonals remain fixed. From this fact, as in item (a), we derive that the cube group is isomorphic to the group of substitutions of four elements (it has order 24), that is, to the symmetric group $S_4$.

(c) Solution. The centres of the faces of the dodecahedron are vertices of an icosahedron. Therefore the dodecahedron and icosahedron groups are isomorphic. For each edge of the icosahedron there is one opposite parallel edge and two pairs of edges perpendicular to it: the edges of one pair begin at the vertices of the faces adjacent to the given edge, and the edges of the other pair belong to the faces having as vertices the endpoints of the given edge. The edges of one of the pairs are parallel, the edges of distinct pairs are perpendicular to each other. Thus, all 30 edges break down into five systems with six in each system. The edges of one system are either parallel or perpendicular, those of different systems are not parallel and not perpendicular. With each system of edges there is associated an octahedron whose vertices are the midpoints of the edges of the given system. Thus are defined five octahedrons inscribed in the icosahedron. To each rotation of the icosahedron there corresponds a substitution of the five indicated systems of edges (or of the octahedrons that correspond to them). A product of two rotations is associated with the product of the corresponding substitutions. Let us consider all rotations of the icosahedron. They include the identical rotation; 24 rotations about each of the six axes passing through opposite vertices, through the angles $\frac{2\pi}{5}$, $\frac{4\pi}{5}$, $\frac{6\pi}{5}$, and $\frac{8\pi}{5}$, 20 rotations about each of the ten axes that pass through the centres of opposite faces, through the angles $\frac{\pi}{3}$ and $\frac{4\pi}{3}$, 15 rotations about each of the fifteen axes passing through the midpoints of the opposite edges, through the angle $\pi$. The total number of rotations is $1 + 24 + 20 + 15 = 60$. According to the answer to problem 1639 they exhaust all rotations of the icosahedron. A direct verification shows that for each nonidentical rotational rotation there is an edge that is carried by the given rotation into another edge that is neither parallel nor perpendicular to the given edge. For this reason.
rotations through the angles $\frac{2k\pi}{5}$, $k = 1, 2, 3, 4$ about each of the six axes passing through opposite vertices. By a rotation about the vertex $A$ through the angle $\alpha$ we mean a rotation about the axis passing through $A$ and the opposite vertex, through the angle $\alpha$ counterclockwise when looking along the axis from $A$ to the opposite vertex. At each vertex we note one of the plane angles with the given vertex. Each rotation of the icosahedron is fully characterized by indicating the vertex $B$ into which the given vertex $A$ passes. $B$ may coincide with $A$, and the plane angle at $A$, into which the marked angle at $A$ passes. Therefore each rotation $x$ that carries $A$ into $B$ is represented in the form of a product $x = yz$, where $y$ carries the marked angle $\alpha$ into the marked angle $\beta$, and $z$ is a rotation about the vertex $B$ through the angle $\alpha$. The inverse element $x^{-1} = z^{-1}y^{-1}$ is a product of the rotation $x^{-1}$ about $B$ through the angle $-\alpha$ and the rotation $y^{-1}$ that carries the marked angle $B$ into the marked angle $A$. Now suppose $g$ is the rotation about the vertex $A$, and $x$ is any element of the group, which element carries $A$ into $B$. Representing $x$ in the form of a product, $x = yz$, as indicated above, we find that the conjugate element $x^{-1}gy = z^{-1}y^{-1}gyz$ is again a rotation through the angle $\alpha$ but this time about vertex $B$. In particular, if $A$ and $B$ are opposite vertices, then the rotation about $B$ through the angle $\alpha$ coincides with the rotation about $A$ through the angle $2\pi - \alpha$. Thus, all rotations through the angles $\frac{2\pi}{5}$ and $\frac{4\pi}{5}$ belong to one class of conjugate elements, just as all the rotations through the angles $\frac{4\pi}{5}$ and $\frac{6\pi}{5}$. We now show that the rotations $g_1$ and $g_2$ about the vertex $A$ through the angles $\frac{2\pi}{5}$ and $\frac{4\pi}{5}$ belong to distinct classes. If $x$ carries $A$ into a different vertex $B$, then $x^{-1}gy$ is a rotation about $B$ and either will not be a rotation about $A$ or (if $B$ is opposite to $A$) will be a rotation about $A$ through the angle $\frac{2\pi}{5}$, that is, $x^{-1}gy = g_1 \neq g_2$. But if $x$ is a rotation about $A$, then $g_1$ and $x$ are elements of a cyclic (and, hence, commutative) subgroup of rotations about $A$ and again $x^{-1}gy = g_1 \neq g_2$.

Thus, all elements of the fifth order split up into two classes with 42 elements in each class. Similarly, marking one plane angle of each face and one vertex of each edge, we see that 20 elements of the third order (rotations through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the axes passing through the centres of opposite faces) constitute one class and 15 elements of the second order (rotations through the angle $\pi$ about the axes passing through the midpoints of opposite edges) also constitute one class.

(b) The normal divisor should consist of integral classes, should contain the unit element, and its order should divide the order 60 of the icosahedron group. According to item (a), classes of conjugate elements coincide. It is only possible to form two sums such that include the summand 1 and divide the number 60, namely, 1 and 60. This yields only two normal divisors: the unit subgroup and the entire group.

1678. Hint. Use the problems 1662 (c) and 1677 (b).

1681. A homomorphism is fully determined by the image of the generating element $a$. Below are indicated the possible images of that element:

(a) any element of the group; the number of homomorphisms is equal to $p$;

(b) $e$, $b$, $b^2$, $b^3$, $b^4$, $b^5$;

(c) $e$, $b$, $b^2$, $b^3$, $b^4$, $b^5$;

(d) $e$, $b^5$, $b^{14}$; (e) $e$.

1683. (a) A cyclic group $\{e\}$ of the fourth order, where $aq = a^2$;

(b) a cyclic group $\{g\}$ of the second order, where $ag = a^3$;

(c) a field of residues modulo 5;

(d) a ring of residues modulo 6;

(e) a ring of residues modulo $n$.

1685. (a) A cyclic group of order $n$;

(b) a cyclic group of order 5;

(c) a cyclic group of order 6;

(d) a cyclic group of order 2.

1688. Hint. In the cases (d), (e), and (h), consider the mapping $f(z) = z^k$, and in the case (f) consider the mapping $f(z) = z^{1/2}$.

1691. The subgroup $\{(1 2 3)\}$ in group $S_3$ has index 3 but does not contain the element $1/3$ of order 2.

1693. Hint. Assuming that $G/Z$ is a cyclic group, choose element $a$ in the class that serves as the generating element of that group and show that $a$ and $Z$ generate the entire group $G$.

1694. Solution. Use induction on the order $n$ of the group $G$. For $n = 2$ the group $G$ is a cyclic group of the second order, and the theorem holds with respect to it. Let the theorem hold true for all groups whose orders are less than $n$ and let $G$ be a group of order $n$. First let $G$ be commutative. Take any element $a$ that differs from the unit element $e$ of $G$. Its order is $k > 1$. If $k$ is divisible by $p$, $k = pq$, then the element $a^p$ is of order $p$. If $k$ is not divisible by $p$, then the order $n'$ of the factor-group $G' = G/\{a\}$ of $G$ with respect to the cyclic subgroup $\{a\}$ is equal to $\frac{n'}{k} < n$ and is divisible by $p$. By the hypothesis of our induction, $G'$ contains the element $b'$ of order $p$. Let $b$ be an element of $G$ that appears in the coset $a^{-1}b$. From $b'p = e'$, where $e'$ is the unit element of $G'$, it follows that $b'p = a^k$, whence $b'p = a^k = e'$, which is impossible. Hence, $b'p = e$ but $b'p = e'$, i.e., the element $b'$ is divisible by the order of element $p$ of element $a$.

Now let the group $G$ be noncommutative. If there is a subgroup $H$ different from $G$ whose index is not divisible by $p$, then the order $|H|$ of $H$ is less than $n$ and is divisible by $p$. By the induction hypothesis, $H$ contains an element of order $p$. But if the indices of subgroups elements conjugate to any element of $G$ not appearing in its centre $Z$.

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(problem 1664) is divisible by \( p \) (problem 1671). Since the order \( n \) of the group \( G \) is also divisible by \( p \), it follows that the order of the centre \( Z \) is divisible by \( p \) and is less than \( n \), since \( G \) is noncommutative. By the induction hypothesis, \( Z \) contains an element of order \( p \).


1701. (a) \( \{a\} = \{3a\} \div \{2a\} \)
(b) \( \{a\} = \{4a\} \div \{6a\} \)
(c) \( \{a\} = \{15a\} \div \{20a\} \div \{12a\} \)
(d) \( \{a\} = \{225a\} \div \{100a\} \div \{36a\} \)

1702. Hint. In the case of (c) take advantage of problem 1700 (b).

1703. Hints. (a) For \( A \) and \( B \) take, respectively, the sets of all elements \( a \) and \( b \) of \( G \) for which \( pa = 0 \) and \( qb = 0 \); (b) consider the decompositions \( n = p_i^{k_i} \ldots p_s^{k_s} \) of order \( n \) of group \( G \) into prime factors, and apply (a).

1704. (a) \( G \) (3); (b) \( G \) (4); \( G \) (2, 2); (c) \( G \) (2, 3); (d) \( G \) (8), \( G \) (2, 4), \( G \) (2, 2, 2); (e) \( G \) (4, 3), \( G \) (2, 2, 2, 2); (f) \( G \) (2, 4, 3), \( G \) (2, 2, 3, 3); (g) \( G \) (4, 9), \( G \) (2, 2, 9), \( G \) (4, 4, 3, 3); (h) \( G \) (2, 2, 2, 3); (i) \( G \) (2, 2, 3, 5); (j) \( G \) (4, 15), \( G \) (2, 2, 9), \( G \) (4, 4, 3, 3); (k) \( G \) (16, 3), \( G \) (2, 2, 9), \( G \) (4, 4, 3, 3); (l) \( G \) (2, 2, 2, 2); (m) \( G \) (8, 9), \( G \) (2, 2, 9), \( G \) (4, 4, 3, 3);

1705. If \( Z \) is a cyclic group of order \( k \) and \( Z \) is an infinite cyclic group, then the desired direct decomposition of the factor-group \( G/H \) has the form:

(a) \( Z + Z + Z_1 \)
(b) \( Z + Z_1 \)
(c) \( Z_1 + Z \)
(d) \( Z + Z_1 \)
(e) \( Z + Z + Z_1 \)
(f) \( Z + Z_1 \)
(g) \( Z + Z_1 \)
(h) \( Z + Z_1 \)
(i) \( Z + Z_1 \)
(j) \( Z + Z_1 \)
(k) \( Z + Z_1 \)
(l) \( Z + Z_1 \)

1706. (c) The group \( G \) can be uniquely decomposed into a direct sum of subgroups: \( G = A_1 + A_2 + \ldots + A_s \), where \( A_i \) is a cyclic subgroup of order \( p_i \). Any nonzero subgroup \( H \) of group \( G \) is a direct sum of certain of the subgroups \( A_i \). The number of all subgroups is equal to \( 2^s \). Hint. Use the item (b) and show that if \( h \) is the generator of the subgroup \( H \), then \( H \) is a direct sum of those subgroups \( A_i \) which contain the nonzero components of the element \( h \).

1707. Hints. (c) To prove the decomposition \( G = H + K \) take any element \( a_i \) outside \( H \), then any element \( a_i \) outside \( \{H, a_i\} \) and so forth, and put \( K = \{a_1, a_2, \ldots\} \).

(d) Any subgroup \( H \) of order \( p^r \) can be decomposed into a direct sum \( S \) of cyclic subgroups of order \( p \). Suppose the decomposition is of the form

\[
H = \{a_1\} + \{a_2\} + \ldots + \{a_t\}.
\]

We find the number of all systems \( \{a_1, a_2, \ldots, a_t\} \) defined in the appropriate manner for all subgroups \( H \) of order \( p^r \). Since \( a_1 \neq 0 \), it follows that for \( a_2 \) we have \( p^k - 1 \) possibilities. Since \( a_2 \neq 0 \), it follows that for \( a_3 \) we have \( p^k - p \) possibilities, and so forth. In similar fashion we find the number of all systems \( \{a_1, a_2, \ldots, a_t\} \) that yield one group \( H \) of order \( p^r \). The number \( n \) of all subgroups of order \( p^r \) is equal to the quotient of the two numbers that were found.

1708. Hint. First consider the case of the primary group, then take the decomposition of the group into primary components (problem 1703 (b)) and apply problem 1700 (b).

1710. A ring.

1711. A ring. For \( n = 0 \) we obtain a zero ring consisting of the single number 0, which is the unit element of the ring and is at the same time its inverse element. The zero ring is a field and a field must contain more than one element.


1724. Matrices with rational \( a, b \), form a field, those with real \( a, b \) form a ring, but not a field.

1725. Polynomials in sines and cosines and polynomials in cosines alone form a ring, but the polynomials in sines alone do not form a ring. Hint. To prove that polynomials in sines do not form a ring, make use of the fact that the product of two odd functions is an even function.

1726. They do not. Hint. Use the irreducibility of the polynomials \( x^n - 2 \) over the field of rational numbers to prove that \( \sqrt{2} + \sqrt{2} = \sqrt{4} \) does not belong to the set under consideration.

1727. \( \frac{1}{43} (5 + 9 \sqrt{2} - \sqrt{6}) \). Hint. To prove uniqueness, take advantage of the irreducibility of the polynomial \( x^n - 2 \) over the field of rational numbers. Use the method of undetermined coefficients to seek the inverse element.

1728. \( x^{-3} = \frac{1}{198} (19 - \sqrt{5} - 11 \sqrt{25}) \).

1729. Hint. Use the property of an irreducible polynomial of being relatively prime to any polynomial of lower degree.

1730. \( \beta^{-1} = \frac{1}{405} (101 + 37 \alpha + 4 \alpha^2) \).

Hint. If \( q(x) = x^n - x + 3 \), then use the method of undetermined coefficients to find the polynomials \( f_1(x) \) of first degree and \( q(x) \) of second degree that satisfy the equation \( f_1(x) q(x) - \beta (x) = 1 \) and set \( x = \alpha \) in that equation.

1732. For example, \( f_1(x) = \{0 \text{ for } x \leq 0, \text{ for } x > 0, \}

1734. The zero divisors are of the form \( \{a, 0\} \), where \( a \neq 0 \), \( 0, b \), where \( b \neq 0 \).

1736. The matrices in which the element in the upper left-hand corner is nonzero will not be left divisors of zero but will be right divisors of zero.

1738. Hint. Remove the brackets in the product \( (a + b) (a + c) \) in two different ways.
1740. Matrices of order \( n > 2 \) with elements taken from the given field form a ring with several left units, that all rows beginning with the second consist of zeros; and under a similar condition for columns, they form a ring with several right units.

1742. Hint. Let \( a \) be a nonzero element of a ring. Show that the correspondence \( x \rightarrow ax \), where \( x \) is any element, is a one-to-one mapping of the given ring onto itself.


1747. Hint. Find the matrices \( \Phi, I, J, K \) that correspond to the unit elements \( 1, i, j, k \) and verify the multiplication table for \( \Phi \):

\[ \Phi \Phi = I, \quad \Phi J = -J, \quad \Phi K = K, \quad \Phi I = I \]

1749. Two such automorphisms are possible: the identical automorphism and the automorphism that carries each number into its conjugate.

1750. Hint. Show that any number field contains the number 1, the integers, and, finally, fractions.

1751. Hint. Consider the images of unity, the integers, and of fractions.

1752. Hint. Show that a positive number, being the square of a real number, passes into a positive number. Then, taking advantage of the fact that there is a rational number between two distinct real numbers and also using the preservation of rational numbers, prove the invariability of any real number.

1753. Only two such mappings are possible: the identical mapping and a mapping that carries any complex number into its conjugate.

1757. The system modulo 5 is inconsistent, but the system modulo 7 has a unique solution: \( x = 2, y = 6, z = 5 \).

1758. (a) \( x + 2 \), (b) \( 1750 \). (a) \( 1 \), (b) \( 5x + 1 \).

1759. (a) \( x \neq 2, x + 2, 1 \). 1760. (a) \( x \neq 2, x + 2, 1 \).

1761. (a) Solution. Suppose that \( f(x) \) and \( g(x) \) have a common divisor \( d(x) \) of positive degree over the field of rational numbers. Then \( f(x) = a(x) d(x) \), \( g(x) = b(x) d(x) \), where \( a(x), b(x), d(x) \) are polynomials with rational coefficients. Taking out the common denominators and the largest common divisor \( m \) of the numerators of the coefficients and applying the Gaussian lemma on the product of primitive polynomials, we obtain:

\[ f(x) = a_1 \cdots a_n, \quad g(x) = b_1 \cdots b_n \]

The degree of \( d(x) \) is equal to the degree of \( d'(x) \) and the leading coefficient of \( d_1(x) \) is not divisible by \( p \). Passing to the field of residues modulo \( p \), we obtain the largest divisor of positive degree for \( f(x) \) and \( g(x) \) over that field, but that is impossible.

(c) The polynomials \( f(x) = x \), \( g(x) = x + p \) are relatively prime over the field of rational numbers and are equal to \( x \), that is, they are not relatively prime over the field of residues modulo \( p \).

1762. Hint. If \( f(x) \) and \( g(x) \) are relatively prime, then, by obtaining polynomials with integral coefficients and \( c \) is an integer, prove that \( f(x) \) and \( g(x) \) are relatively prime over the field of residues modulo \( p \) for any \( c \) that does not divide \( c \).

1763. \( x + 1 \) and \( x + 2 \). 1764. \( x + 3 \) and \( x + 4 \).

1769. \( f_1 = x^2 + 1, \quad f_2 = x^2 + x + 2, \quad f_3 = x^2 + 2x + 1 \).

1771. Hint. Make use of the Gaussian lemma, and from the factorization of \( f(x) \) into two factors with rational coefficients obtain a factorization into two factors with integral coefficients. The polynomial \( f(x) = px^2 + (p + 1)x + 1 \) is reducible over the field of rational numbers, but with respect to modulus \( p \) it is equal to \( x + 1 \) and, hence, is irreducible.

1772. Hint. Assume the contrary, apply a decomposition of the group \( G \) into a direct product of primary cyclic subgroups, make use of problem 1700 (c) and the theorem that the equation \( x^n = 1 \) does not have more than \( n \) distinct roots in the field \( \mathbb{Z}_p \).

1773. Solution. First prove the lemma from group theory. If two elements \( a, b \) of a cyclic group \( G \) are not squares, then their product is a square.

The set \( H \) of elements taken from \( G \) that are squares is a subgroup. The factor group \( G/H \) is cyclic. If \( c = cH \) is its generator, then \( c^2 = c^2H = H \). Hence, either \( H = \{e, cH\} \) is a group of the second order and \( ab \in cH, bH = H \), that is, \( ab \) is a square.

From this it follows that one of the numbers \( 2, 3 \), \( 0 \) is a square with respect to any prime number modulus \( p \) indeed, for \( p = 2 \) we have \( 2 = 0 \), for \( p = 3 \) we also have \( 3 = 0 \). If \( p > 3 \), then 2 and 3 may be regarded as elements of the multiplicative group \( G \) of a field of residues modulo \( p \). By problem 1772 the group \( G \) is cyclic and, by the lemma that was proved above, if 2 and 3 are not squares, then \( 2 \cdot 3 = 0 \) is a square.

The polynomial

\[ f(x) = (x - \sqrt{2} + \sqrt{3}) (x + \sqrt{2} - \sqrt{3}) (x^2 - 2\sqrt{2} - \sqrt{3}) (x + \sqrt{2} + \sqrt{3}) \]

is irreducible over the field of rational numbers since the linear factors and their products taken two at a time are not polynomials with rational coefficients.
The members of a finite set of numbers \( S \) with operations \( + \) and \( \cdot \) over itself, \( A \) and \( b \) are zero divisors for each \( a, b \) \( \in S \); for the given \( M > 0 \) in \( \mathbb{Z} \) there only exists a finite set of numbers \( z \) with \( N (z) < M \); the only unit divisors are \( 1 \); the divisor \( z \) with least norm exceeding 1 is prime.

1780. Hint. By transformation of the irreducible element into a polynomial of degree exceeding two.

1781. (a) An ideal, (b) a subring, (c) an ideal, (d) it is a subring of the additive group, (e) a subring, (f) an ideal, (g) a subring, (h) it is a subgroup of the additive group.

1782. Hint. Show that any ideal \( I \) is generated by its nonzero element \( a \) that is smallest in the following meaning: (a) in absolute value; (b) in degree; (c) with respect to modulus. In each case make use of the existence of division (with a remainder) by the element \( b \neq 0 \); the remainder is either equal to zero or is less than the divisor in the sense indicated above.

1783. If \( a \neq 0 \) for any \( a \neq 0 \) in \( Z \), then \( \varphi \) is an isomorphism and \( \varphi (Z) \) is isomorphic to \( Z \). If \( a \neq 0 \) for some \( a \neq 0 \) taken from \( Z \), and \( \varphi \) is the smallest positive number for which \( \varphi (2) = 0 \), then \( \varphi (2) \) is isomorphic to \( \mathbb{Z} \).

1784. Hint. Show that any cosets consisting of the numbers \( b + c \) with the following properties: (1) \( a + b = 0 \), (2) \( a + b = \) odd, (3) \( a + b = \) odd and \( b + c = \) odd; (4) \( a + b = \) odd; (c) the coset \( H \) containing \( 1 + i \) is a zero divisor, and \( H^2 = 0 \).

1785. The number of elements is equal to \( p^0 \).

1786. Hint. Use problem 1647 (b).

1787. (b) Let \( a \) and \( b \) be two distinct elements of second order \( \mathbb{Z} \) isomorphic to \( Z \). If \( a \neq 0 \) for some \( a \neq 0 \) taken from \( Z \), and \( \alpha \) is the smallest positive number for which \( \alpha (2) = 0 \), then \( \alpha (2) \) is isomorphic to \( \mathbb{Z} \).

1788. Hint. Consider the union of the modules \( M_1 \).

1789. Hint. Show that \( b \neq 1 \); \( b \) is a homomorphism mapping \( H \) onto \( (A + H)/A \) and use the theorem on homomorphisms for modules.

1790. Hint. Use induction on \( a \). For \( a = 1 \) use problem 1815 and 1818. For \( a > 1 \) assume that in \( M \) there is an infinite increasing chain of distinct submodules \( M_1 \leq M_2 \leq \ldots \) set \( A = \{ 2 \} \) and \( M = M_1 \).

1791. If \( a \neq 0 \) for any \( a \neq 0 \) in \( Z \), then \( \varphi \) is an isomorphism and \( \varphi (Z) \) is isomorphic to \( Z \). If \( a \neq 0 \) for some \( a \neq 0 \) taken from \( Z \), and \( \varphi \) is the smallest positive number for which \( \varphi (2) = 0 \), then \( \varphi (2) \) is isomorphic to \( \mathbb{Z} \).

1792. Hint. Prove by induction that, in the linear relation, set \( m = 1, 2, 3, \ldots, m \).

1793. Hint. In the cases (c), (d), (e) differentiate twice and with respect to moduli.

1794. Hint. Use the Vandermonde determinant.

1795. The dimension is equal to \( C_{n+1}^1 - 1 = C_n \).

1796. Hint. Take all monomials for the basis and with each monomial of the form \( x_{i_1}^{1_{i_1}} \cdots x_{i_n}^{1_{i_n}} \) associate the row

\[
\begin{pmatrix}
1 & x_{i_1} & 1_{i_1} & \cdots & 1 & x_{i_n} & 1_{i_n}
\end{pmatrix}
\]

and find a basis for the preceding problem.

1797. (b) The dimension of \( L \) is equal to \( n \); (c) the dimension of \( L_1 \) is \( n - k - 1 \); (d) \( L \) is not a subspace.

1798. Hint. When proving necessary, obtain the equality \( h = CA \), where \( C \) is a nonsingular matrix made up of the coefficients in the expressions of the system (2) in terms of (1) When proving sufficient, apply the condition (3) to the coordinates of the vector \( b \), and, when computing the rank by the bordering method, show that the rank of the resulting matrix is equal to \( k \).

1799. (i) Substituting in the plane \( x_{i_j}^y, L = Ox, M = Op, \) and \( N = Os \), and \( P = Oq \), the straight lines that pass through the coordinate origin and is distinct from the axes \( q \), is a projection on \( L \) parallel to \( M \), and \( q, \) as a projection on \( L \) parallel to \( N \). Then \( q = q \).

The condition (3) is satisfied for \( P \) and \( q \) and \( r \) are coplanar.
Hints. (b) Show that if \( q_1 \) and \( q_2 \) are idempotent, then \( q_1 + q_2 \) is idempotent; if and only if \( q_1 q_2 = q_2 q_1 = 0 \). Multiplying this equality on the left and on the right by \( q_1 \), prove that it is equivalent to condition (1). (c) Using (a), reduce (c) to (b). (d) From \( \phi x = x \), derive \( \phi x = \phi_2 x = x \). Then use the representation \( x = \phi x + \langle x - \phi x \rangle \). 1837. Hint. Consider \((\phi (x_1 + x_2), y)\) and \((\phi (\lambda x), y)\).

1838. \( \frac{1}{5} \sqrt{10} \). 1839. If \( L \) is the subspace of all vectors, in each of which only a finite number of coordinates are nonzero, then

\[ L^* = 0, \quad L^* = V \quad (L^*)^* = V \neq L. \]

1841. Let \( A \) be the transformation matrix in an orthonormal basis. (a) A rotation of the plane through some angle about the coordinate origin if \( A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \); a reflection of the plane in some straight line that passes through the origin if \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \); (b) a rotation of space through a certain angle about an axis that passes through the origin if \( A = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \); the above-mentioned rotation followed by a reflection of space in the plane passing through the origin and perpendicular to the axis of rotation if \( A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).

1842. A rotation about the axis defined by the vector \( f = (1, 1, 0) \) through the angle \( \alpha = 60^\circ \) in the negative direction. Hint. We seek the vector \( f \) as an eigenvector that belongs to the eigenvalue 1. We find the angle of rotation \( \alpha \) from the condition \( \cos \alpha + (x^2 + y^2 + z^2) = 1 \) obtained from the invariance of the trace of the transformation matrix \( \phi \). To determine the direction of rotation we take a vector that does not lie on the axis of rotation, say \( e_1 \), its image \( \phi e_1 \), and we seek the sign of the determinant from the coordinates of these three vectors, that is, we seek the orientation of the triplet of vectors \( e_1, \phi e_1, f \).

1843. (a) The zero transformation; (b) a rotation through the angle \( \pi/2 \) in the positive or negative direction followed by a multiplication by a nonnegative number; (c) \( g x = a \times x \). When \( a \neq 0 \), the transformation \( g \) reduces to projecting the vector \( x \) on the plane perpendicular to the vector \( a \), to the rotation about \( a \) through the angle \( \pi/2 \) in the positive direction, and to multiplication by the length \( |a| \). Hint. Consider the transformation matrix \( \phi \) in the orthonormal basis \( A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) and set \( a = (a_1, a_2, a_3) \), where \( a_1 = -a_2, a_3 = -a_3, \lambda = 0. \)

1844. Hint. Find the basis \( e_1, e_2, \ldots, e_n \) for which

\[ l(x_1) = 1, \quad l(e_3) = \ldots = l(e_n) = 0. \]

1845. Hint. Assume that \( l_1(a) \neq 0, l_2(b) \neq 0 \) and consider the vector \( a + b \).

1849. Hint. Prove that if \( b(x, y) \neq 0 \), then

\[ \frac{l_1(x)}{l_2(x)} = \frac{l_1(y)}{l_2(y)} = \lambda \neq 0. \]

Consider the product \( (x - \lambda d_0(x)) l_2(x) \) and apply problem 1848. 1850. Hint. Use the problems 1851 and 1849. 1851. Hint. Use problem 1848.

1852. Hint. Set \( y = -\frac{l(x)}{l(a)} \).

1853. Hint. For nonzero functions, take the vector \( a \) not lying in \( S \), set \( \lambda = l_1(a) \), and apply problem 1852 (b).

1854. (a) A one-sheet hyperboloid; (b) a two-sheet hyperboloid. Hint. Pass to homogeneous coordinates.

1856. If \( e_1, \ldots, e_n \) is a normal basis (in which \( f(x) \) can be written as a quadratic form of normal aspect) and \( f(e_j) = 1 \) \( (i = 1, 2, \ldots, n) \), then for the desired basis \( e_1, \ldots, e_n \), set \( e_j = \lambda e_j \) \( (i = 1, \ldots, n) \), and apply problem 1852 (b). 1857. Hint. Use the problems 1856 and 1856.

1858. Hint. Consider the case (b) under the condition \( p \leq q \).

Taking the notation \( f(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \) for the form of the normal aspect, verify that \( K \) contains the subspace \( L \) given by the equations

\[ x_1 - x_{p+1} = 0, \ldots, x_{p} - x_{p+q} = 0, \]

| 1/2 | 0, \ldots, 0 |

1859. (a) \[ \sum_{j=1}^{n} \lambda_j z_j = 0 \] (i = 1, 2, \ldots, n) \( q \), then assume that \( K \) contains a subspace \( L \) of dimension \( n - q \) given by the equations

\[ z_1 = 0, \ldots, z_p = 0, \quad x_{p+q+1} = 0, \ldots, x_n = 0 \]

and obtain a contradiction.

1859. (a) Solution. If \( f(x) \) is written in the normal form in a suitable basis, then the equation of the surface \( S \) can be written thus:

\[ x_1^2 + \cdots + x_p^2 - x_{p+1} - \cdots - x_{p+q}^2 = 0. \]
If $p = 0$, then there are no real points on $S$. Here, $\min (p + 1, q) = -1$. The theorem holds if the dimension of the empty set is taken to be equal to $-1$.

Let $p > 0$. Then there are points on $S$, for example, $(1, 0, \ldots, 0)$, that is, zero-dimensional manifolds. Let $P'$ be a manifold of maximal dimension $k$ appearing in $S$. $P'$ is given by a system of $n - k$ linearly independent equations

$$
\sum_{j=1}^{n} a_{ij}x_j = b_i \quad (i = 1, 2, \ldots, n-k).
$$

This system cannot be homogeneous since the zero solution does not satisfy equation (1). For the sake of simplicity, we assume that the determinant $d$, of the order $n - k$ and made up of coefficients of the first $n - k$ unknowns, is nonzero. Then the general solution of system (2) may be written down thus:

$$
x_i = c_i + \sum_{j=1}^{n-k} e_{ijn}x_j + a_{ij} \quad (i = 1, 2, \ldots, n_k).
$$

Consider the $(n + 1)$-dimensional space $V_{n+1}$. In it we take any basis and assume that $V_{n}$ is spanned by the first $n$ vectors of the basis.

Consider the homogeneous system of equations

$$
\sum_{j=1}^{n} a_{ij}x_j = 0 \quad (i = 1, 2, \ldots, n-k).
$$

The coefficients are the same as in (2). The general solution is of the form

$$
x_i = c_i + \sum_{j=1}^{n-k} e_{ijn}x_j + a_{ij} \quad (i = 1, 2, \ldots, n-k).
$$

Next consider the cone $K$ specified by the equation

$$
x_1^2 + \cdots + x_n^2 = 0.
$$

We will prove that the $(k + 1)$-dimensional subspace $L$ given by system (4) lies in the cone $K$. Any solution of system (4) in which $x_n = 1$ yields, after $x_n$ is dropped, the solution of system (2), that is, a vector in $P' \subseteq S$. But such a solution satisfies equation (4) and, hence, lies in $K$.

If $x$ is any solution of system (4), in which $x_n = 0$, then $x \in K$, whence $x \in K$. Let $x = (x_1, \ldots, x_n)$ be a solution of system (4) with $x_n = 0$. There is a solution $x'$ of system (4) in which the free unknowns have the values

$$
x_l - x_n = \frac{c_{l-n}}{c_{n-1}}, \quad x_n = \alpha_n, \quad x_{n+1} = \frac{1}{\alpha_n} (l = 1, 2, \ldots).
$$

From the formulas (5) it is clear that $\lim_{t \to \infty} x' = x$. From what has been proved above, $x' \in K$. Passing to the limit in (6) after substitution into the coordinates $x'$, as $t \to \infty$, we obtain $x \in K$.

Thus, $L \subseteq K$. The indices of inertia of $K$ are equal to $p, q + 1$. According to problem 1858, $k + 1 \leq \min (p + q + 1)$, whence $k \leq \min (p - 1, q)$.

Let $K'$ be a cone with the equation

$$
x_1^2 + \cdots + x_{n-k}^2 = 0,
$$

where the signs coincide with the signs of the corresponding terms of equation (1). By problem 1858, the cone $K'$ contains the subspace $L_k$ of dimension $k = \min (p - 1, q)$ lying in the subspace with the equation $x_1 = 0$. In $V_n$, take the vector $a_0 = (1, 0, \ldots, 0)$. Then $S$ contains the manifold $P' \subseteq L'$ of dimension $k = \min (p - 1, q)$. The proof of assertion (a) is complete.

(b) Hint. Let $1 < r < n$, reduce (b) to (a) in an $r$-dimensional space.

1860. (a) $1$; (b) $0$; (c) $1$; (d) $1$; (e) $1$; (f) $1$; (g) $1$; (h) $n$; (i) $1$; (j) $1$; (k) $1$; (l) $1$.

1861. (b) Hint. Find the systems of linear equations that specify the left and right kernels.

1862. The basis $L_0$ is formed by the vector $(3, -1, \ldots)$, and the basis $L_0^*$ by the vector $(2, -1)$.

1863. (a) Hint. Let $1 < r < n$. Take a function whose matrix, in a certain basis, has in the upper left-hand corner a nonsingular square block of order $r$ that is neither symmetric nor skew-symmetric, and is elsewhere zero.

1864. Hint. Make use of the zero subspace of the function $f(x, y)$.

1865. Hint. Show that the coordinates of all vectors in $L_0$ satisfy the matrix equation $AY = 0$, where $A$ is the matrix of $b(x, y)$ in a certain basis of the space $P_0$. $A$ is a matrix whose rows involve the coordinates of any basis of the subspace $L$ in the given basis of $V_n$, and $Y$ is the column of coordinates of the vector $y \in L_0$ in the same basis.

1866. First proof. Since $b(x, y) = 0$ but $b(x, x) = 0$, it follows that there are vectors $x_1, x_2$ such that $b(x_1, x_2) = 0$. Multiplying one of these vectors by $1/b(x_1, x_2)$, we obtain the vectors $e_1, e_2$ for which $b(e_1, e_2) = 1$. The vectors $e_1, e_2$ are linearly independent, since if $e_2 = ae_2$, then $b(e_1, e_2) = b(e_1, ae_2) = 0$. Let $L_0$ be a two-dimensional subspace spanned by $e_1, e_2$, and let $L_0$ be the set of all $y \in V_n$ such that $b(x, y) = 0$ for any $x \in L$. By problem 1858, $L_0$ is a subspace of dimension $n - 2$. The intersection of $L_0$ and $L_0$ contains only the zero vector, for if $x \in L_0$, then $x = x_1 + x_2 + x_3$. If $x \in L_0$, then $b(x, x_1) = b(x, x_1, y) = 0$ and $b(x, e_2) = b(x, e_2, y) = 0$; $b(e_1, e_2) = b(e_1, e_2, y) = 1$, whence $\alpha_1 = 0, \alpha_2 = 0, x_3 = 0$. By problem 1296, the dimension of $L_0$ is $n - 2$. From the direct sum $L_0 \oplus L_0$, we also have $L_0 \oplus L_0$, and there are vectors $e_3, e_4, e_5 \in L_0$ for which $b(e_3, e_4) = 1$ and so forth.

Second proof. This proof affords a practical method for finding the nonsingular linear transformation of the unknowns, which transposes.
formation reduces the given bilinear form to the canonical form given in the problem.

Suppose, in a certain basis, \( b(x, y) = \sum_{i,j} a_{ij}x_iy_j \). By changing the numbering of the unknowns we may assume that \( a_{12} \neq 0 \). Write the form as

\[
b(x, y) = x_1(a_{12}y_2 + \ldots + a_{1n}y_n) - y_1(a_{12}x_2 + \ldots + a_{1n}x_n) + b_1(x, y)\]

and perform a nonsingular transformation of the unknowns

\[
x'_1 = x_1, x'_2 = a_{12}x_2 + \ldots + a_{1n}x_n, x'_3 = x_3, \ldots, x'_n = x_n
\]

and the same transformation of \( y_i \). This yields

\[
b(x, y) = x'_1y'_2 - x'_2y'_1 + b_2(x, y).
\]

If \( b_2(x, y) \) does not contain \( x'_2, y'_1 \), then we proceed in a similar fashion. Otherwise \( b_2(x, y) = \sum_{j=2}^n a_{ij}x'_jy'_j \), where \( a_{ij} \neq 0 \) for a certain \( i, 2 \leq k \leq n \). Having performed a nonsingular transformation of the unknowns

\[
x'_1 = x'_1 - a_{21}x'_2 - \ldots - a_{n1}x'_n, x'_2 = x'_2, \ldots, x'_n = x'_n
\]

and a similar transformation of \( y'_j \), we obtain

\[
b(x, y) = x'_1y'_2 - x'_2y'_1 + b_2(x, y)
\]

If \( b_2(x, y) \neq 0 \), we proceed in a similar manner.

1867. \( b(x, y) = u_2v_2 - u_1v_2 - u_1v_1 - u_2v_1, u_1 = x_1 + 3x_4, u_2 = x_2 + 2x_3, x_3 = 2x, u_4 = 6x_4, \) and the same expression of \( v_i \) in terms of \( y_i \).

1883. \( b(x, y) = u_2v_2 - u_1v_2 - u_1v_1 - u_2v_1, u_1 = x_1 + 4x_3, u_2 = x_2 + 2x_3, u_3 = 2x, u_4 = 8x_4, \) and the same expression of \( v_i \) in terms of \( y_i \).

In matrix terms we obtain the following assertion: for a real symmetric matrix \( A \) to be orthogonally similar to a matrix in which all the elements of the main diagonal are zero, it is necessary and sufficient that the trace of \( A \) be zero.

**Hint.** When proving sufficiency, use induction on \( n \). For \( n > 1 \) take any orthonormal basis \( f_1, f_2, \ldots, f_n \). If it does not lie on the cone, then prove that there exist vectors \( f_1, f_2 \) for which \( \langle f_1, f_2 \rangle > 0 \), \( \langle f_2, f_2 \rangle < 0 \), and for the first vector of the desired basis take \( f_1 \) obtained by normalizing the vector \( f_1 + \lambda f_2 \), where \( \lambda \) is found from the condition \( \langle f_1 + \lambda f_2, f_1 \rangle = 0 \).

1870. **Hint.** Use the property: four distinct points form a parallelogram if and only if their radius vectors satisfy the condition \( x_1 + x_2 = x_3 + x_4 \).

1875. \( x_1 = -4 + 3t_1 - 4t_2, x_2 = 2 - t_1 + t_2, x_3 = t_1, x_4 = t_2, x_5 = 3 - 4t_3, x_6 = 0, x_7 = t_4 \).

1877. \( 3x_1 - x_2 - x_3 = 8, x_1 - 2x_2 \mid x_4 = 3, 5x_1 - 2x_7 - x_8 = 7 \).

1878. \( 3x_1 - 3x_2 - x_3 + 2 = 0, 2x_1 - 3x_2 - x_3 + 7 = 0, x_1 - 5x_2 - x_3 + 8 = 0 \).

Let \( r \) and \( r' \) be, respectively, the ranks of the matrices

\[
A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix}
\]

and \( A' = \begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \\ d_4 & e_4 & f_4 \end{pmatrix} \)

and let \( r_{12} \) and \( r'_{12} \) be, respectively, the ranks of the matrices of the 1st and 2nd rows of the matrices \( A \) and \( A' \).

The following five cases are possible for which the indicated values of the ranks are necessary and sufficient:

1. three planes pass through one point: \( r = r' = 3 \);
2. the three planes do not have any points in common but intersect pairwise along straight lines (they form a prism): \( r = r' = 2 \);
3. two planes are parallel and the third plane intersects them: \( r_{12} = 1, r = r' = r_{12} = 2 \), \( r = 3 \) and two similar cases;
4. the three planes pass through one straight line: \( r = r' = 3 \);
5. the three planes are parallel: \( r = 1, r'_{12} = r'_{15} = r_{20} = 2 \).

Let \( r \) and \( r' \) be, respectively, the ranks of the matrices

\[
\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}
\]

and

\[
\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}
\]

Four cases are possible:

1. \( r = 3, r' = 4 \); the straight lines are skew to each other;
2. \( r = 2, r' = 3 \); the lines intersect; (3) \( r = 2, r' = 3 \); the lines are parallel; (4) \( r = r' = 2 \); the lines coincide.

1884. Let \( r \) and \( r' \) be, respectively, the ranks of a simple matrix and an augmented matrix of the combined systems of equations (1) and (2). Five cases are possible:

1. \( r = r' = 4 \); the planes intersect in one point; (2) \( r = 3 \); the planes are skew to each other and are parallel to the right-hand members; (3) \( r = r' = 3 \); the lines are linearly independent along a straight line; (4) \( r = 2 \); the planes intersect; (5) \( r = r' = 2 \); the planes coincide.

1885. Let \( r \) and \( r' \) be, respectively, the ranks of the matrices

\[
\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}
\]

and

\[
\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}
\]

Three cases are possible:
(1) \( r = r' = 2 \); the hyperplanes intersect along an \((n - 2)\)-dimensional plane; (2) \( r = 1, r' = 2 \); that is \( \frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} + \frac{c}{d} \); the hyperplanes are parallel; (3) \( r = r' = 1 \), the hyperplanes coincide.

1886. Hint. Apply problem 1874 and the two-dimensional property of subspaces.

1888, 1889, and 1890. Hint. Use problem 1887.

1891. If \( \pi_1 \parallel \pi_2 \), then \( \pi_3 = \pi_1 \); if \( \pi_1 \nparallel \pi_2 \), then \( \pi_3 = a_1 + + (L_1 + L_2) \).

1893. Let the plane \( \pi_i \) be specified by the system of equations

\[
\begin{align*}
    a_{i1}x_1 + \ldots + a_{in}x_n &= c_i \\
    a_{i1}x_1 + \ldots + a_{in}x_n &= c_i
\end{align*}
\]

and let the plane \( \pi_n \) be specified by the system of equations

\[
\begin{align*}
    b_{n1}x_1 + \ldots + b_{nn}x_n &= d_1 \\
    b_{n1}x_1 + \ldots + b_{nn}x_n &= d_1
\end{align*}
\]

Furthermore, let \( r_1, r_2 \) and \( r_3, r_4 \) be, respectively, the rank of the matrix made up of the coefficients of the unknowns and the rank of the augmented matrix of the systems (1) and (2); \( r \) and \( r' \) are, respectively, the rank of the matrix of the coefficients of the unknowns and the rank of the augmented matrix of the combined system consisting of all equations of systems (1) and (2). For the systems (1) and (2) to specify planes, it is necessary and sufficient that each of them be consistent, that is, \( r_1 = r_2 \) and \( r_3 = r_4 \). When these conditions are complied with, it is necessary and sufficient for the parallelism of the given planes that \( r = \max (r_1, r_2) \), \( r' = \min (r_3, r_4) \).

1895. (a) The polyhedron \( P \) is given by the system of inequalities

\[
\begin{align*}
    z_1 &\geq 0, \\
    z_2 &\geq 0, \\
    z_3 &\geq 0, \\
    z_4 &\geq 0, \\
    z_1 + z_2 &\leq 1, \\
    z_3 + z_4 &\leq 1
\end{align*}
\]

(b) The three-dimensional faces are four quadrangular pyramids: \( OABCD \) with vertex \( D \), \( OABCE \) with vertex \( E \), \( ODEFA \) with vertex \( A \), \( ODEFB \) with vertex \( B \), and the four tetrahedrons \( ACDF, ACEF, BCDF, BCEF \). Hint. Pass a three-dimensional plane through every four points not lying on one two-dimensional plane. If \( \sum a_ix_i = b \) is the equation of such a plane and for the coordinates of all the given points we have either \( \sum a_ix_i \geq b \) or \( \sum a_ix_i < b \), then the corresponding inequality enters into a system of inequalities that specify the polyhedron \( P \). The convex closure of all points lying in the given three-dimensional plane will be a three-dimensional face of the given polyhedron. For example, the points \( O, A, B, D \) determine a three-dimensional plane with the equation \( x_4 = 0 \). For all given points, \( x_4 \geq 0 \). Therefore the inequality \( x_4 \geq 0 \) enters into the desired system.

The five given points \( O, A, B, C, D \) lie on the three-dimensional plane \( x_4 = 0 \). Their convex closure is a pyramid which is a three-dimensional face of the polyhedron \( P \). On the contrary, for the four points \( A, B, C, D \) determine a three-dimensional plane with the equation \( x_4 = z_4 = 0 \); for the point \( D \) we have \( x_3 = z_4 > 0 \) and for the point \( E \) we have \( x_3 = z_4 < 0 \). Thus, this plane does not lead to the desired inequality and does not contain the faces of the polyhedron \( P \). In order to reduce the number of quadruplets of points under consideration, take into account that the two quadruplets \( OABC \) and \( ODEF \) are of an equal status and lie in two-dimensional planes.

1896. (a) The polyhedron \( P \) is given by the system of inequalities

\[
\begin{align*}
    x_1 &\geq 0, \\
    x_2 &\geq 0, \\
    x_3 &\geq 0, \\
    x_4 &\geq 0, \\
    x_1 + x_2 &\leq 1, \\
    x_3 + x_4 &\leq 1
\end{align*}
\]

(b) The three-dimensional faces are: the cube \( OABCDEF \) and six quadrangular pyramids with a common vertex \( H: OBCFH, OACEH, ODFGH, ADEGH, BDFGH, CEGFH \). Hint. Use problem 1887.

1897. Five vertices \( A(1, 1, 1), B(1, 1, -2), C(4, -2, 1), D(-2, 1, 1), E(1, 1, -2) \). The polyhedron has six triangular faces \( ABC, ABD, ACD, BCE, BDE, CDE \) and constitutes two tetrahedrons \( ABCD \) and \( BCDE \) with the common base \( BCD \).

1898. (a) A tetrahedron with the vertices \((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\);

(b) an octahedron with the vertices \((1, 1, 0, 0), (1, 0, 1, 0), (0, 1, 1, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0)\);

(c) a triangular prism with bases at the points \((1, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0)\);

(d) a square with the vertices \((1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)\).

1899. (a) 6, (b) 28, (c) 14, (d) 2, (e) 4, (f) 4, (g) 7, (h) 6, (i) 3, (j) 3, (k) 1, (l) 6, (m) 5, and (n) 12. Hint. Introduce a system of coordinates and consider the parametric equations of the straight line and the plane in vector form.

1901. (a) \( a_1 = \psi (e) \) \((i = 1, 2, \ldots, n)\);

1902. \( F (\psi) = \psi \) \(\in V_n^2\).

1903. \( A' = C^*A \), where \( C \) is the transition matrix from the old basis to the new basis, written down in columns.

1905. \( A' = D^*A \), where \( D = (C^*)^{-1} \), and \( C \) is the transition matrix from the old basis to the new basis written down in columns.

1906. \( A' = C^{-1}AC = D^*AC \), where \( C \) is the transition matrix from the old basis to the new basis written down in columns, and \( D = (C^*)^{-1} \).

1907. (b) \( F (x, \psi) = \psi (x) \), \( x \in V_n^2 \), \( \psi \in V_n^2 \).

1908. Hint. (b) Take the contraction \( a_{ik}a_{ij} \) of the tensor \( a_{ij}^b \), where \( b \) is the tensor with components \( b_{ij} = a_{ij}^b \) in one basis.

1913. \( a_{11}^i a_{22}^i \ldots a_{nn}^i = n! a_{i1}^i a_{i2}^i \ldots a_{in}^i \).

1914. \( a_{ij}^i \ldots a_{im}^i = (-1)^{st} \), where \( s \) is the number of inversions in the permutation \( i_1, i_2, \ldots, i_n \) and \( t \) is the number of inversions in the permutation \( j_1, j_2, \ldots, j_n \) if the upper and lower indices are distinct; otherwise the indicated coordinate is zero.

1917. It is an invariant equal to the number \( 0 \) in all bases.

1918. Hint. (a) Verify this for each of the equivalence rules indicated in the introduction to the section; (b) in order to prove necessity for \( x \neq 0 \), take the contraction with respect to \( \psi \) with the property \( \psi (x') \neq 0 \). To prove sufficiency, put \( n = 0x \) for the pair \( x o' \) in the pair.
When passing to a new basis with the same orientation, the quantities \( b_{ij} \) vary as the components of a twice covariant tensor. When passing to a basis with opposite orientation, the quantities \( b_{ij} \) change sign as well.

228. \( a_{ij} = g_{ij} e^{ijk} \) \((i, j, k = 1, 2, \ldots, n)\). Hint. Contract both sides of the equality given in the problem with \( g_{ij} e^{ijk} \) with respect to \( i \) and \( j \), make use of the relation \( g_{ij} g_{ij} = \delta_{ij} \), and then change the notation \( i, j \) to \( \alpha, \beta \) and \( \alpha', \beta' \) to \( i, j \).

228. \( S = \frac{3}{2}, \ h = 1 \). Hint. Seek the area via the formula \( S = \frac{1}{2} bc \sin \theta \) or make use of problem 1935.

229. \( Q (-4, 2, 0) \). Hint. Write the parametric equations of the straight line \( PQ \) in contravariant components.

1930. \( H \) int. When proving (b), take the orthonormal basis \( e_1, \ e_2, \ e_3, \ e_4 \) where \( e_i \) is directed along \( u_i \) and \( e_2, \ e_3, \ e_4 \) lie in the same three-dimensional plane as \( x, \ y, \ z \) and have the same orientation. Make use of the expression of the oriented volume via formulas (17) and (18) of the introduction to this section.

231. The invariant equal to the number \( n \) in any basis. 1933. 6.

1934. (b) \( G = \begin{pmatrix} 6 & -5 & 2 \\ -5 & 5 & -2 \\ 2 & -2 & -1 \end{pmatrix} \), (c) \( \begin{pmatrix} 2 \\ \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \end{pmatrix} \) to within sign.

1935. \( H \) int. First method: take the given vectors for the basis; second method: choose an orthonormal basis.

1937. \( d = \sqrt{a^2 + b^2 + c^2} \). Hint. Use problem 1936.

1938. \( H \) int. Pass to an orthonormal basis with the same orientation.
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